

An Automated Deduction of the Foulis-Holland Theorem in Quantum Logic

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Abstract

A quantum logic (QL) is a logic of a system of propositions describing the outcome of measurements of quantum systems. These propositions can be cast in the form "Measurand X had (didn't have) value V at time t". In classical physics, the system of such propositions can be cast in a form that has a Boolean logic (BL). A consequence of the existence of non-commuting observables in quantum mechanics is that QL, in contrast, does not satisfy the Boolean distributive law. To help understand the structural relationships between BL and QL, quantum logicians have paid much attention to "quasi"-distributive theorems, which identify non-trivial conditions under which the distributive law holds in QLS. One of the better known quasi-distributive theorems is the Foulis-Holland Theorem (FHT). Automated deduction frameworks (ADFs) have continued to yield insight into QLS in general and into the FHT in particular. Here, using the ADF prover9, I provide an automated deduction of the FHT. The derivation appears to be novel.

Keywords: automated deduction, quantum computing, orthomodular lattice, Hilbert space

1.0 Introduction

We can think of the set of propositions describing the outcome of measurements of physical systems as a set of propositions of the form "Measurand X had (didn't have) value V at time t" (Birkhoff and von Neumann 1936) where X ranges over physical measurands (e.g., mass, velocity, charge, ...), together with logical connectives among these propositions. Equivalently, we can cast such propositions as answers to questions of the form "Did measurand X have value V at time t?". These propositions are called "yes/no" propositions (YNPs; Jauch 1968).

In classical physics, the system of YNPs has a Boolean logic (Jauch 1968), and that logic is isomorphic to a Boolean lattice (Jauch 1968, 80; Birkhoff 1967, Chap. I, Section 10).

In quantum mechanics (Messiah 1958), the system of YNPs can be cast in a form that is isomorphic to the algebra (Warner 1990, Chap. VI, Section 33), $C(H)$, of the system of closed linear subspaces of (equivalently, to the system of linear operators on (observables in) a Hilbert space H (von Neumann 1936, Akhiezer and Glazman 1961, Megill and Pavičić 2001, Pavičić and Megill 1998). This algebra is a model (Chang and Keisler 2012) of an ortholattice (OL; Birkhoff 1967, Section 14). An OL can thus be thought of as a kind of "quantum logic" (QL; Dalla Chiara et al. 2018). $C(H)$ is also a model of an orthomodular lattice (OML; Birkhoff 1967; Beran 1985), which is an OL conjoined with the orthomodularity axiom (OMA (Beran 1985, Chap. II, Section 3)).

The non-commutativity of (certain pairs of) observables (e.g., position and momentum (Messiah 1958, esp. Chap. 4, Sections 12-14)) implies that the Boolean distributive law (Warner 1990, Ex. 16.21 and Ex. 21.24)

$$(D) \quad (x \wedge (y \vee z)) = (x \wedge y) \vee (x \wedge z)$$

does not hold in QL (Jauch 1968, Section 5-4). A QL, in fact, can be thought of as a BL in which (D) does not hold.

Figure 1 shows the (equational) formulation (McCune and Padmanabhan 1996) of the axioms of QL used in this paper.

Lattice axioms

$$\begin{aligned} x &= c(c(x)) && (AxLat1) \\ x \vee y &= y \vee x && (AxLat2) \\ (x \vee y) \vee z &= x \vee (y \vee z) && (AxLat3) \\ (x \wedge y) \wedge z &= x \wedge (y \wedge z) && (AxLat4) \\ x \vee (x \wedge y) &= x && (AxLat5) \\ x \wedge (x \vee y) &= x && (AxLat6) \end{aligned}$$

Orthogonality axioms

$$\begin{aligned} c(x) \wedge x &= 0 && (AxOL1) \\ c(x) \vee x &= 1 && (AxOL2) \\ x \wedge y &= c(c(x) \vee c(y)) && (AxOL3) \end{aligned}$$

Orthomodularity axiom

$$y \vee (c(y) \wedge (x \vee y)) = x \vee y \quad (AxOM)$$

where

x, y are variables ranging over lattice nodes
 \wedge is lattice meet
 \vee is lattice join
 $c(x)$ is the orthocomplement of x
 $=$ is equality (Gries and Schneider 1993, Section 1.3)
 1 is the maximum lattice element ($= x \vee c(x)$)
 0 is the minimum lattice element ($= c(1)$)

Figure 1. Lattice, orthogonality, and orthomodularity axioms of the QL used in this paper.

Because non-commutativity of some observables is arguably the most fundamental difference between the respective logics of YNPs in classical and quantum physics (Cohen 1989, Def. 3B.14), "quasi"-distributive QL theorems, i.e., QL theorems that (non-trivially) identify conditions under which distributivity holds, have been of significant interest to

quantum logicians. Among the better known of the quasi-distributive theorems is the Foulis-Holland Theorem (Holland 1995), shown in Figure 2.

Let $C(x,y)$ be defined as $(x = ((x \wedge y) \vee (x \wedge c(y))))$. Let " \rightarrow " be first-order implication (see Church 1956). Then

Foulis-Holland theorem FH1

$$(C(x,y) \ \& \ C(x,z)) \ \rightarrow \ ((x \wedge (y \vee z)) = ((x \wedge y) \vee (x \wedge z)))$$

Foulis-Holland theorem FH2

$$(C(x,y) \ \& \ C(x,z)) \ \rightarrow \ ((y \wedge (x \vee z)) = ((y \wedge x) \vee (y \wedge z)))$$

Foulis-Holland theorem FH3

$$(C(x,y) \ \& \ C(x,z)) \ \rightarrow \ ((x \vee (y \wedge z)) = ((x \vee y) \wedge (x \vee z)))$$

Foulis-Holland theorem FH4

$$(C(x,y) \ \& \ C(x,z)) \ \rightarrow \ ((y \vee (x \wedge z)) = ((y \vee x) \wedge (y \vee z)))$$

Figure 2. The Foulis-Holland Theorem (FHT). Note that the hypotheses of FH1 - FH4 are identical. Each of the variables x , y , and z is universally quantified across each of FH1 – FH4, i.e., the scope of the universal quantifiers of FH1, FH2, FH3, and FH4, is FH1, FH2, FH3, and FH4, respectively. Within any of FH1 – FH4, x , y , and z are distinct. The theorem is stated here as the conjunction of four propositions to facilitate traceability to the stepwise implementation of the proof of FHT in Section 3.0.

Semi-automated deduction frameworks have provided insight into QLs in general. Megill 2019, for example, contains proofs of ~1000 OL/OML theorems, including a proof of the FHT, *verified* but not *generated* by the *Metamath* proof-checking software. Fully automated deduction frameworks (ADFs) have been used to derive FHT-related theorems. Horner 2005, for example, uses the *Otter* (McCune 2003) ADF to generate a proof of FH1. Horner 2018 contains a *Mathematica*-generated proof of the Marsden-Herman Theorem (Marsden and Herman 1974), which is closely related to the Foulis-Holland Theorem.

2.0 Method

prover9 (McCune 2011) is an ADF for first-order theories (Chang and Keisler 2012, Section 1.4). The axioms of Figure 1 were implemented in *prover9* scripts shown in Appendices 1 – 4, configured to derive the FHT. The scripts were executed under *prover9* running under Windows 10/Cygwin, on a Dell Inspiron 545 containing an Intel Q8200 quadprocessor clocked at 2.33 GHz and containing 8 GB memory.

3.0 Results

Figures 3-6 depict summaries of *prover9* derivations¹ of FH1 – FH4. In Figures 3-6, a percent (%) mark at the beginning of a line delimits a comment. For the purpose of this paper, a *prover9* statement is any non-blank sequence of characters that does not begin with a % sign and which ends with a period (some statements may span more than one text line due to line wrap in these Figures). The *prover9* statements shown in Figures 3-6 are first-order sentences (Church 1956). “ \leftrightarrow ” (if and only if), “ \rightarrow ” (implies), “&” (and), “|” (or), “=” (equality), “!=” (not identical), and “-“ (not) are standard first-order connectives. In any line in Figures 2-5, x, y, and z are distinct first-order variables. All variables in all *prover9* sentences are (implicitly) universally quantified, unless otherwise explicitly quantified.

Strings of the form “# label (...)” are tag-phrases that *prover9* transmits in various ways through a *prover9* analysis. The content of such tags can be formulated in such a way that they help facilitate traceability of the use of the tagged statement in a derivation. *prover9* derivations are by default proofs by contradiction.

In Figures 3-6, the text in square brackets ([]) immediately following a statement compactly describes the inference rules *prover9* uses to derive that statement. The derivations in Figure 3-6 use four inference rules. Informally stated, these are:

- a. rewrite and back_rewrite: substitute for a term t1 a term t2 that is equal to t1 (Baader and Nipkow 1998, Section 3.1)
- b. flip: interchange the LHS and RHS of an identity.
- c. paramodulation (abbrev. “para”). Let C[r] be a clause containing a term r at some designated position. If t is another term, we let C[t] be the results of replacing this single designated occurrence of r by t (Quaife 1992, 15; Kalman 2001, Chap. 14).
- d. assume: given a theory (Chang and Keisler 2012, Section 1.4) T (e.g., the theory shown in Figure 1), we can assume any sentence in T.
- e. resolve: from “not-A or B” and “not-A”, infer B (Leitsch 1997)

Figures 3-6 are generated entirely from the theory shown in Figures 1 and 2, with one exception. The proofs introduce a definition, “Def. 2.20” (see, for example, Line 18 in Figure 3; adapted from Megill and Pavičić 2001). This definition, which does not affect the

¹ See Kalman 2001 (esp. Chaps. 1-14) for a good overview of the theory of *prover9*. See McCune 2011 for further detail on *prover9* notation, syntax, and semantics.

implicational scope of Figure 1, is introduced solely to shorten the derivations that *prover9* generates.

----- begin derivation of FH1

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===== PROOF =====
% Proof 1 at 1.30 (+ 0.11) seconds: "Foullis-Holland Theorem 1".
% Length of proof is 52.
% Level of proof is 13.
% Maximum clause weight is 29.000.
% Given clauses 151.
2 C(x,y) & C(x,z) -> x ^ (y v z) = (x ^ y) v (x ^ z) # label("Foullis-
Holland Theorem 1") # label(non_clause) # label(goal). [goal].
7 x = c(c(x)) # label("AxL1"). [assumption].
8 c(c(x)) = x. [copy(7),flip(a)].
9 x v y = y v x # label("AxL2"). [assumption].
10 (x v y) v z = x v (y v z) # label("AxL3"). [assumption].
12 x v (x ^ y) = x # label("AxL5"). [assumption].
13 x ^ (x v y) = x # label("AxL6"). [assumption].
14 c(x) ^ x = 0 # label("AxOL1"). [assumption].
15 c(x) v x = 1 # label("AxOL2"). [assumption].
16 x v c(x) = 1. [copy(15),rewrite([9(2)])].
17 x ^ y = c(c(x) v c(y)) # label("AxOL3"). [assumption].
18 1_2 = x v ((y ^ c(x)) v (c(y) ^ c(x))) # label("Df. 2.20").
[assumption].
19 x v (c(y v x) v c(c(y) v x)) = 1_2.
[copy(18),rewrite([17(3),8(4),17(7),8(6),8(6),9(7)]),flip(a)].
20 x v (c(x) ^ (y v x)) = y v x # label("OMLaw"). [assumption].
21 x v c(x v c(y v x)) = y v x. [copy(20),rewrite([17(3),8(2)])].
22 (c1 ^ c2) v (c1 ^ c3) != c1 ^ (c2 v c3) # label("Foullis-Holland Theorem
1") # answer("Foullis-Holland Theorem 1"). [deny(2)].
23 c(c(c1) v c(c2 v c3)) != c(c(c1) v c(c2)) v c(c(c1) v c(c3)) #
answer("Foullis-Holland Theorem 1").
[copy(22),rewrite([17(3),17(9),17(18)]),flip(a)].
28 c(1) = 0. [back_rewrite(14),rewrite([17(2),8(2),16(2)])].
29 c(c(x) v c(x v y)) = x. [back_rewrite(13),rewrite([17(2)])].
30 x v c(c(x) v c(y)) = x. [back_rewrite(12),rewrite([17(1)])].
32 x v (y v z) = y v (x v z). [para(9(a,1),10(a,1,1)),rewrite([10(2)])].
34 x v (y v c(x v y)) = 1. [para(16(a,1),10(a,1)),flip(a)].
35 x v (c(x v y) v c(c(y) v x)) = 1_2. [para(9(a,1),19(a,1,2,1,1))].
44 x v c(x v c(x v y)) = y v x. [para(9(a,1),21(a,1,2,1,2,1))].
49 1_2 = 1.
[para(21(a,1),19(a,1,2,1,1)),rewrite([29(13),9(7),9(8),16(8)]),flip(a)].
60 x v (c(x v y) v c(c(y) v x)) = 1. [back_rewrite(35),rewrite([49(8)])].
68 c(x) v c(x v y) = c(x). [para(29(a,1),8(a,1,1)),flip(a)].
81 x v c(c(x) v y) = x. [para(8(a,1),30(a,1,2,1,2))].
85 x v 0 = x. [para(16(a,1),30(a,1,2,1)),rewrite([28(2)])].
86 x v c(y v c(x)) = x. [para(21(a,1),30(a,1,2,1))].
104 0 v x = x. [para(85(a,1),9(a,1)),flip(a)].
126 x v (y v c(c(x) v z)) = y v x. [para(81(a,1),32(a,1,2)),flip(a)].
149 c(x) v c(y v x) = c(x). [para(8(a,1),86(a,1,2,1,2))].
164 c(x) v (y v c(x v z)) = y v c(x). [para(68(a,1),32(a,1,2)),flip(a)].
216 c(x v y) v c(y v c(x v y)) = c(y).
[para(149(a,1),21(a,1,2,1,2,1)),rewrite([8(6),9(5),149(11)])].
217 c(x) v (y v c(z v x)) = y v c(x). [para(149(a,1),32(a,1,2)),flip(a)].

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456 x v c(x v c(y v c(x v y))) = 1.
[para(34(a,1),60(a,1,2,1,1)),rewrite([28(2),9(6),104(8)])].
579 c(x v y) v c(x v c(c(y) v z)) = c(x v c(c(y) v z)).
[para(126(a,1),149(a,1,2,1)),rewrite([9(8)])].
1418 c(x v y) v (z v c(x v (u v y))) = z v c(x v y).
[para(32(a,1),217(a,1,2,2,1))].
3051 x v c(y v c(x v y)) = x.
[para(456(a,1),44(a,1,2,1)),rewrite([28(2),85(2),9(5)]),flip(a)].
3077 x v c(y v x) = x v c(y).
[para(3051(a,1),86(a,1,2,1)),rewrite([9(5),164(5)]),flip(a)].
3285 c(x v y) v c(y v c(x)) = c(y).
[back_rewrite(216),rewrite([3077(5)])].
3552 x v c(y v (z v x)) = x v c(y v z). [para(10(a,1),3077(a,1,2,1))].
3553 x v (y v c(z v x)) = y v (x v c(z)).
[para(3077(a,1),32(a,1,2)),flip(a)].
3554 x v (y v c(x v z)) = x v (y v c(z)).
[para(32(a,1),3077(a,1,2,1)),rewrite([10(5),3552(4),10(7)])].
3561 c(x v y) v c(c(y) v x) = c(x).
[para(60(a,1),3077(a,1,2,1)),rewrite([28(8),9(8),104(8),9(14),217(14),9(10),68(10)])].
3979 c(x v c(y v z)) = c(x v c(z)) v c(x v (z v c(y))).
[para(217(a,1),3285(a,1,1,1)),rewrite([8(8),9(7),3553(7)]),flip(a)].
4175 c(x v c(y)) v c(x v (y v z)) = c(x v c(y)) v c(x v z).
[back_rewrite(579),rewrite([3979(7),8(7),1418(10),3979(11),8(11)]),flip(a)].
4176 c(c3 v (c(c1) v c(c2))) v c(c(c1) v c(c3)) != c(c(c1) v c(c2)) v
c(c(c1) v c(c3)) # answer("Foulis-Holland Theorem 1").
[back_rewrite(23),rewrite([3979(8),32(13),9(15)])].
4204 c(x v c(y v z)) = c(x v c(z)) v c(x v c(y)).
[back_rewrite(3979),rewrite([4175(12)])].
4234 c(x v (y v c(z))) v c(y v c(x)) = c(y v c(z)) v c(y v c(x)).
[para(126(a,1),3561(a,1,2,1)),rewrite([8(2),9(4),3554(4),8(10),4204(12)])]
.
4235 $F # answer("Foulis-Holland Theorem 1"). [resolve(4234,a,4176,a)].
===== end of proof =====

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Figure 3. Summary of *prover9* proof of FH1.

----- end derivation of FH1

Figure 3a is brief explanation of representative lines of the derivation of FH1 (Figure 3). (The derivations shown in Figures 4-6 are highly similar to the derivation of FH1.)

The first five lines are comments containing identification and performance information.

The remaining lines in the proof begin with a Line Number.

“LHS” means “left-hand-side”. “RHS” means “right-hand-side”. $t1 \leftarrow t2$ means “substitute term $t2$ for term $t1$ ”.

Line 2. State FH1 (state (but do not assume)) of the proposition to be proved. That proposition is denied in Line 22 for the purpose of proof by contradiction).

Line 7. Assume axiom AxL1.

Line 8. Interchange the LHS and RHS of Line 7 (equality is symmetric (Gries and Schneider 1993, Section 1.3)

Lines 9 - 15. Assume axioms AxL2, AxL3, AxL5, AxL6, AxOL1, and AxOL2.

Line 16. Interchange the terms on the LHS of Line 15, by substitution of those terms into Line 9.

...

Line 18. Assume Def. 2.20.

...

Line 22. Deny the proposition to be proved (see Line 2).

...

Line 29.

Given Line 13, $x \wedge (x \vee y) = x$. The LHS of Line 13 has the same form as the LHS Line 17. In Line 17, $x \leftarrow x$ and $y \leftarrow (x \vee y)$. This yields Line 29.

...

Line 4235. The conjunction of Lines 4234 and 4176 is a contradiction. Therefore, by proof by contradiction, the denial (at Line 22) of the sentence to be proved (Line 2) is false; Line 2, the proposition to be proved, follows. Q.E.D.

Figure 3a. A brief explanation of representative statements in Figure 3.

----- begin derivation of FH2

===== PROOF =====
% Proof 1 at 2.95 (+ 0.12) seconds: "Foullis-Holland Theorem 2".
% Length of proof is 59.

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% Level of proof is 11.
% Maximum clause weight is 29.000.
% Given clauses 238.
2 C(x,y) & C(x,z) -> y ^ (x v z) = (y ^ x) v (y ^ z) # label("Foulis-
Holland Theorem 2") # label(non_clause) # label(goal). [goal].
7 x = c(c(x)) # label("AxL1"). [assumption].
8 c(c(x)) = x. [copy(7),flip(a)].
9 x v y = y v x # label("AxL2"). [assumption].
10 (x v y) v z = x v (y v z) # label("AxL3"). [assumption].
12 x v (x ^ y) = x # label("AxL5"). [assumption].
13 x ^ (x v y) = x # label("AxL6"). [assumption].
14 c(x) ^ x = 0 # label("AxOL1"). [assumption].
15 c(x) v x = 1 # label("AxOL2"). [assumption].
16 x v c(x) = 1. [copy(15),rewrite([9(2)])].
17 x ^ y = c(c(x) v c(y)) # label("AxOL3"). [assumption].
18 1_2 = x v ((y ^ c(x)) v (c(y) ^ c(x))) # label("Df. 2.20").
[assumption].
19 x v (c(y v x) v c(c(y) v x)) = 1_2.
[copy(18),rewrite([17(3),8(4),17(7),8(6),8(6),9(7)]),flip(a)].
20 x v (c(x) ^ (y v x)) = y v x # label("OMLaw"). [assumption].
21 x v c(x v c(y v x)) = y v x. [copy(20),rewrite([17(3),8(2)])].
22 (c2 ^ c1) v (c2 ^ c3) != c2 ^ (c1 v c3) # label("Foulis-Holland Theorem
2") # answer("Foulis-Holland Theorem 2"). [deny(2)].
23 c(c(c2) v c(c1 v c3)) != c(c(c1) v c(c2)) v c(c(c2) v c(c3)) #
answer("Foulis-Holland Theorem 2").
[copy(22),rewrite([17(3),9(5),17(9),17(18)]),flip(a)].
28 c(1) = 0. [back_rewrite(14),rewrite([17(2),8(2),16(2)])].
29 c(c(x) v c(x v y)) = x. [back_rewrite(13),rewrite([17(2)])].
30 x v c(c(x) v c(y)) = x. [back_rewrite(12),rewrite([17(1)])].
32 x v (y v z) = y v (x v z). [para(9(a,1),10(a,1,1)),rewrite([10(2)])].
34 x v (y v c(x v y)) = 1. [para(16(a,1),10(a,1)),flip(a)].
35 x v (c(x v y) v c(c(y) v x)) = 1_2. [para(9(a,1),19(a,1,2,1,1))].
37 x v (c(y v x) v (c(c(y) v x) v z)) = 1_2 v z.
[para(19(a,1),10(a,1,1)),rewrite([10(9)]),flip(a)].
44 x v c(x v c(x v y)) = y v x. [para(9(a,1),21(a,1,2,1,2,1))].
46 x v (y v c(x v (y v c(z v (x v y)))))) = z v (x v y).
[para(21(a,1),10(a,1)),rewrite([10(7)]),flip(a)].
49 1_2 = 1.
[para(21(a,1),19(a,1,2,1,1)),rewrite([29(13),9(7),9(8),16(8)]),flip(a)].
58 x v (c(y v x) v (c(c(y) v x) v z)) = 1 v z.
[back_rewrite(37),rewrite([49(9)])].
60 x v (c(x v y) v c(c(y) v x)) = 1. [back_rewrite(35),rewrite([49(8)])].
68 c(x) v c(x v y) = c(x). [para(29(a,1),8(a,1,1)),flip(a)].
72 c(0 v c(x)) = x. [para(16(a,1),29(a,1,1,2,1)),rewrite([28(3),9(3)])].
76 1 v x = 1. [para(28(a,1),29(a,1,1,1)),rewrite([72(6)])].
78 x v (c(y v x) v (c(c(y) v x) v z)) = 1.
[back_rewrite(58),rewrite([76(10)])].
81 x v c(c(x) v y) = x. [para(8(a,1),30(a,1,2,1,2))].
85 x v 0 = x. [para(16(a,1),30(a,1,2,1)),rewrite([28(2)])].
86 x v c(y v c(x)) = x. [para(21(a,1),30(a,1,2,1))].
92 x v (y v c(x v c(z v x))) = y v (z v x).
[para(21(a,1),32(a,1,2)),flip(a)].
104 0 v x = x. [para(85(a,1),9(a,1)),flip(a)].
124 x v (c(c(x) v y) v z) = x v z. [para(81(a,1),10(a,1,1)),flip(a)].
149 c(x) v c(y v x) = c(x). [para(8(a,1),86(a,1,2,1,2))].
163 c(x v y) v c(x v (y v z)) = c(x v y). [para(10(a,1),68(a,1,2,1))].
164 c(x) v (y v c(x v z)) = y v c(x). [para(68(a,1),32(a,1,2)),flip(a)].

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175 c(c(x) v y) v (z v x) = z v x.
[para(81(a,1),46(a,1,2,2,1,2,2,1,2)),rewrite([124(10),92(9),81(9)])].
215 c(x) v (c(y v x) v z) = c(x) v z. [para(149(a,1),10(a,1,1)),flip(a)].
216 c(x v y) v c(y v c(x v y)) = c(y).
[para(149(a,1),21(a,1,2,1,2,1)),rewrite([8(6),9(5),149(11)])].
217 c(x) v (y v c(z v x)) = y v c(x). [para(149(a,1),32(a,1,2)),flip(a)].
456 x v c(x v c(y v c(x v y))) = 1.
[para(34(a,1),60(a,1,2,1,1)),rewrite([28(2),9(6),104(8)])].
739 c(x) v (y v (c(z v x) v u)) = c(x) v (y v u).
[para(175(a,1),124(a,1,2,1,1)),rewrite([10(6),10(9)])].
3051 x v c(y v c(x v y)) = x.
[para(456(a,1),44(a,1,2,1)),rewrite([28(2),85(2),9(5)]),flip(a)].
3077 x v c(y v x) = x v c(y).
[para(3051(a,1),86(a,1,2,1)),rewrite([9(5),164(5)]),flip(a)].
3285 c(x v y) v c(y v c(x)) = c(y).
[back_rewrite(216),rewrite([3077(5)])].
3553 x v (y v c(z v x)) = y v (x v c(z)).
[para(3077(a,1),32(a,1,2)),flip(a)].
3566 c(x v y) v (c(c(x) v y) v z) = c(y) v z.
[para(78(a,1),3077(a,1,2,1)),rewrite([28(9),9(9),104(9),9(16),739(16),215(
12)])].
3979 c(x v c(y v z)) = c(x v c(z)) v c(x v (z v c(y))).
[para(217(a,1),3285(a,1,1,1)),rewrite([8(8),9(7),3553(7)]),flip(a)].
4176 c(c3 v (c(c1) v c(c2))) v c(c(c2) v c(c3)) != c(c(c1) v c(c2)) v
c(c(c2) v c(c3)) # answer("Foulis-Holland Theorem 2").
[back_rewrite(23),rewrite([3979(8),9(13),10(13),9(15)])].
4221 c(x v y) v c(y v (z v x)) = c(x v y).
[para(9(a,1),163(a,1,2,1)),rewrite([10(4)])].
5460 c(x v (y v z)) v c(z v x) = c(z v x).
[para(4221(a,1),9(a,1)),flip(a)].
6544 c(x v (y v z)) v c(z v c(x)) = c(y v z) v c(z v c(x)).
[para(5460(a,1),3566(a,1,2))].
6545 $F # answer("Foulis-Holland Theorem 2"). [resolve(6544,a,4176,a)].
===== end of proof =====

```

Figure 4. Summary of *prover9* proof of FH2.

----- begin derivation of FH3

```

===== PROOF =====
% Proof 1 at 1.31 (+ 0.08) seconds: "Foulis-Holland Theorem 3".
% Length of proof is 67.
% Level of proof is 13.
% Maximum clause weight is 30.000.
% Given clauses 151.
2 C(x,y) & C(x,z) -> x v (y ^ z) = (x v y) ^ (x v z) # label("Foulis-
Holland Theorem 3") # label(non_clause) # label(goal). [goal].
7 x = c(c(x)) # label("AxL1"). [assumption].
8 c(c(x)) = x. [copy(7),flip(a)].
9 x v y = y v x # label("AxL2"). [assumption].
10 (x v y) v z = x v (y v z) # label("AxL3"). [assumption].
12 x v (x ^ y) = x # label("AxL5"). [assumption].
13 x ^ (x v y) = x # label("AxL6"). [assumption].

```

```

14  $c(x) \wedge x = 0$  # label("AxOL1"). [assumption].
15  $c(x) \vee x = 1$  # label("AxOL2"). [assumption].
16  $x \vee c(x) = 1$ . [copy(15),rewrite([9(2)])].
17  $x \wedge y = c(c(x) \vee c(y))$  # label("AxOL3"). [assumption].
18  $1_2 = x \vee ((y \wedge c(x)) \vee (c(y) \wedge c(x)))$  # label("Df. 2.20").
[assumption].
19  $x \vee (c(y \vee x) \vee c(c(y) \vee x)) = 1_2$ .
[copy(18),rewrite([17(3),8(4),17(7),8(6),8(6),9(7)]),flip(a)].
20  $x \vee (c(x) \wedge (y \vee x)) = y \vee x$  # label("OMLaw"). [assumption].
21  $x \vee c(x \vee c(y \vee x)) = y \vee x$ . [copy(20),rewrite([17(3),8(2)])].
22  $c1 \vee (c2 \wedge c3) \neq (c1 \vee c2) \wedge (c1 \vee c3)$  # label("Foulis-Holland Theorem
3") # answer("Foulis-Holland Theorem 3"). [deny(2)].
23  $c(c(c1 \vee c2) \vee c(c1 \vee c3)) \neq c1 \vee c(c(c2) \vee c(c3))$  # answer("Foulis-
Holland Theorem 3"). [copy(22),rewrite([17(4),17(15)]),flip(a)].
28  $c(1) = 0$ . [back_rewrite(14),rewrite([17(2),8(2),16(2)])].
29  $c(c(x) \vee c(x \vee y)) = x$ . [back_rewrite(13),rewrite([17(2)])].
30  $x \vee c(c(x) \vee c(y)) = x$ . [back_rewrite(12),rewrite([17(1)])].
32  $x \vee (y \vee z) = y \vee (x \vee z)$ . [para(9(a,1),10(a,1,1)),rewrite([10(2)])].
34  $x \vee (y \vee c(x \vee y)) = 1$ . [para(16(a,1),10(a,1)),flip(a)].
35  $x \vee (c(x \vee y) \vee c(c(y) \vee x)) = 1_2$ . [para(9(a,1),19(a,1,2,1,1))].
44  $x \vee c(x \vee c(x \vee y)) = y \vee x$ . [para(9(a,1),21(a,1,2,1,2,1))].
45  $x \vee (c(x \vee c(y \vee x)) \vee z) = y \vee (x \vee z)$ .
[para(21(a,1),10(a,1,1)),rewrite([10(2)]),flip(a)].
46  $x \vee (y \vee c(x \vee (y \vee c(z \vee (x \vee y)))) = z \vee (x \vee y)$ .
[para(21(a,1),10(a,1)),rewrite([10(7)]),flip(a)].
49  $1_2 = 1$ .
[para(21(a,1),19(a,1,2,1,1)),rewrite([29(13),9(7),9(8),16(8)]),flip(a)].
60  $x \vee (c(x \vee y) \vee c(c(y) \vee x)) = 1$ . [back_rewrite(35),rewrite([49(8)])].
68  $c(x) \vee c(x \vee y) = c(x)$ . [para(29(a,1),8(a,1,1)),flip(a)].
72  $c(0 \vee c(x)) = x$ . [para(16(a,1),29(a,1,1,2,1)),rewrite([28(3),9(3)])].
76  $1 \vee x = 1$ . [para(28(a,1),29(a,1,1,1)),rewrite([72(6)])].
81  $x \vee c(c(x) \vee y) = x$ . [para(8(a,1),30(a,1,2,1,2))].
85  $x \vee 0 = x$ . [para(16(a,1),30(a,1,2,1)),rewrite([28(2)])].
86  $x \vee c(y \vee c(x)) = x$ . [para(21(a,1),30(a,1,2,1))].
92  $x \vee (y \vee c(x \vee c(z \vee x))) = y \vee (z \vee x)$ .
[para(21(a,1),32(a,1,2)),flip(a)].
98  $x \vee (y \vee c(y \vee x)) = 1$ . [para(9(a,1),34(a,1,2,2,1))].
104  $0 \vee x = x$ . [para(85(a,1),9(a,1)),flip(a)].
124  $x \vee (c(c(x) \vee y) \vee z) = x \vee z$ . [para(81(a,1),10(a,1,1)),flip(a)].
149  $c(x) \vee c(y \vee x) = c(x)$ . [para(8(a,1),86(a,1,2,1,2))].
150  $x \vee (c(y \vee c(x)) \vee z) = x \vee z$ . [para(86(a,1),10(a,1,1)),flip(a)].
162  $c(x) \vee (c(x \vee y) \vee z) = c(x) \vee z$ . [para(68(a,1),10(a,1,1)),flip(a)].
164  $c(x) \vee (y \vee c(x \vee z)) = y \vee c(x)$ . [para(68(a,1),32(a,1,2)),flip(a)].
175  $c(c(x) \vee y) \vee (z \vee x) = z \vee x$ .
[para(81(a,1),46(a,1,2,2,1,2,2,1,2)),rewrite([124(10),92(9),81(9)])].
180  $x \vee (y \vee (c(y \vee x) \vee z)) = 1$ .
[para(98(a,1),10(a,1,1)),rewrite([76(2),10(5)]),flip(a)].
215  $c(x) \vee (c(y \vee x) \vee z) = c(x) \vee z$ . [para(149(a,1),10(a,1,1)),flip(a)].
216  $c(x \vee y) \vee c(y \vee c(x \vee y)) = c(y)$ .
[para(149(a,1),21(a,1,2,1,2,1)),rewrite([8(6),9(5),149(11)])].
217  $c(x) \vee (y \vee c(z \vee x)) = y \vee c(x)$ . [para(149(a,1),32(a,1,2)),flip(a)].
219  $c(x \vee y) \vee (c(y \vee c(x \vee y)) \vee z) = c(y) \vee z$ .
[para(149(a,1),45(a,1,2,1,1,2,1)),rewrite([8(6),9(5),215(13)])].
456  $x \vee c(x \vee c(y \vee c(x \vee y))) = 1$ .
[para(34(a,1),60(a,1,2,1,1)),rewrite([28(2),9(6),104(8)])].
611  $c(x \vee y) \vee c(c(z \vee c(x)) \vee y) = c(c(z \vee c(x)) \vee y)$ .
[para(150(a,1),149(a,1,2,1)),rewrite([9(8)])].

```

```

739 c(x) v (y v (c(z v x) v u)) = c(x) v (y v u).
[para(175(a,1),124(a,1,2,1,1)),rewrite([10(6),10(9)])].
1418 c(x v y) v (z v c(x v (u v y))) = z v c(x v y).
[para(32(a,1),217(a,1,2,2,1))].
3051 x v c(y v c(x v y)) = x.
[para(456(a,1),44(a,1,2,1)),rewrite([28(2),85(2),9(5)]),flip(a)].
3077 x v c(y v x) = x v c(y).
[para(3051(a,1),86(a,1,2,1)),rewrite([9(5),164(5)]),flip(a)].
3284 c(x v y) v (c(y v c(x)) v z) = c(y) v z.
[back_rewrite(219),rewrite([3077(5)])].
3285 c(x v y) v c(y v c(x)) = c(y).
[back_rewrite(216),rewrite([3077(5)])].
3550 x v (c(y v x) v z) = x v (c(y) v z).
[para(3077(a,1),10(a,1,1)),rewrite([10(3)]),flip(a)].
3553 x v (y v c(z v x)) = y v (x v c(z)).
[para(3077(a,1),32(a,1,2)),flip(a)].
3560 x v (c(x v y) v z) = c(y) v (x v z).
[para(180(a,1),3077(a,1,2,1)),rewrite([28(6),9(6),104(6),9(10),739(10)])].
3561 c(x v y) v c(c(y) v x) = c(x).
[para(60(a,1),3077(a,1,2,1)),rewrite([28(8),9(8),104(8),9(14),217(14),9(10),68(10)])].
3975 c(c(x v y) v z) = c(c(x) v z) v c(c(y) v (x v z)).
[para(162(a,1),3285(a,1,1,1)),rewrite([8(8),9(7),3560(7)]),flip(a)].
3979 c(x v c(y v z)) = c(x v c(z)) v c(x v (z v c(y))).
[para(217(a,1),3285(a,1,1,1)),rewrite([8(8),9(7),3553(7)]),flip(a)].
4043 c(c(x) v y) v c(z v (x v y)) = c(c(x) v y) v c(z v y).
[back_rewrite(611),rewrite([3975(7),8(7),1418(10),3975(11),8(11)]),flip(a)].
4132 c(c3 v c(c1)) v (c(c2 v (c(c1) v c(c3))) v c(c(c2) v c(c3))) != c1 v c(c(c2) v c(c3)) # answer("Foulis-Holland Theorem 3").
[back_rewrite(23),rewrite([3979(10),9(7),3979(8),9(5),9(13),10(13),9(15),9(24),10(24),68(23),9(21)])].
4192 c(c(x v y) v z) = c(c(x) v z) v c(c(y) v z).
[back_rewrite(3975),rewrite([4043(12)])].
4235 c(x v (c(y) v z)) v c(c(x) v z) = c(c(y) v z) v c(c(x) v z).
[para(150(a,1),3561(a,1,2,1)),rewrite([8(2),9(4),3550(4),8(10),4192(12)])].
.
4241 $F # answer("Foulis-Holland Theorem 3").
[back_rewrite(4132),rewrite([4235(20),3284(19),8(3)]),xx(a)].
===== end of proof =====

```

Figure 5. Summary of *prover9* proof of FH3.

----- begin derivation of FH4

```

===== PROOF =====
% Proof 1 at 1.22 (+ 0.14) seconds: "Foulis-Holland Theorem 4".
% Length of proof is 50.
% Level of proof is 10.
% Maximum clause weight is 21.000.
% Given clauses 148.
2 C(x,y) & C(x,z) -> y v (x ^ z) = (y v x) ^ (y v z) # label("Foulis-
Holland Theorem 4") # label(non_clause) # label(goal). [goal].

```

```

7 x = c(c(x)) # label("AxL1"). [assumption].
8 c(c(x)) = x. [copy(7),flip(a)].
9 x v y = y v x # label("AxL2"). [assumption].
10 (x v y) v z = x v (y v z) # label("AxL3"). [assumption].
12 x v (x ^ y) = x # label("AxL5"). [assumption].
13 x ^ (x v y) = x # label("AxL6"). [assumption].
14 c(x) ^ x = 0 # label("AxOL1"). [assumption].
15 c(x) v x = 1 # label("AxOL2"). [assumption].
16 x v c(x) = 1. [copy(15),rewrite([9(2)])].
17 x ^ y = c(c(x) v c(y)) # label("AxOL3"). [assumption].
18 1_2 = x v ((y ^ c(x)) v (c(y) ^ c(x))) # label("Df. 2.20").
[assumption].
19 x v (c(y v x) v c(c(y) v x)) = 1_2.
[copy(18),rewrite([17(3),8(4),17(7),8(6),8(6),9(7)]),flip(a)].
20 x v (c(x) ^ (y v x)) = y v x # label("OMLaw"). [assumption].
21 x v c(x v c(y v x)) = y v x. [copy(20),rewrite([17(3),8(2)])].
22 c2 v (c1 ^ c3) != (c2 v c1) ^ (c2 v c3) # label("Foulis-Holland Theorem
4") # answer("Foulis-Holland Theorem 4"). [deny(2)].
23 c(c(c1 v c2) v c(c2 v c3)) != c2 v c(c(c1) v c(c3)) # answer("Foulis-
Holland Theorem 4"). [copy(22),rewrite([17(4),9(11),17(15)]),flip(a)].
28 c(1) = 0. [back_rewrite(14),rewrite([17(2),8(2),16(2)])].
29 c(c(x) v c(x v y)) = x. [back_rewrite(13),rewrite([17(2)])].
30 x v c(c(x) v c(y)) = x. [back_rewrite(12),rewrite([17(1)])].
32 x v (y v z) = y v (x v z). [para(9(a,1),10(a,1,1)),rewrite([10(2)])].
34 x v (y v c(x v y)) = 1. [para(16(a,1),10(a,1)),flip(a)].
35 x v (c(x v y) v c(c(y) v x)) = 1_2. [para(9(a,1),19(a,1,2,1,1))].
44 x v c(x v c(x v y)) = y v x. [para(9(a,1),21(a,1,2,1,2,1))].
46 x v (y v c(x v (y v c(z v (x v y)))))) = z v (x v y).
[para(21(a,1),10(a,1)),rewrite([10(7)]),flip(a)].
49 1_2 = 1.
[para(21(a,1),19(a,1,2,1,1)),rewrite([29(13),9(7),9(8),16(8)]),flip(a)].
60 x v (c(x v y) v c(c(y) v x)) = 1. [back_rewrite(35),rewrite([49(8)])].
68 c(x) v c(x v y) = c(x). [para(29(a,1),8(a,1,1)),flip(a)].
73 c(x v y) v c(x v c(x v y)) = c(x).
[para(29(a,1),21(a,1,2,1,2)),rewrite([9(5),68(11)])].
85 x v 0 = x. [para(16(a,1),30(a,1,2,1)),rewrite([28(2)])].
86 x v c(y v c(x)) = x. [para(21(a,1),30(a,1,2,1))].
92 x v (y v c(x v c(z v x))) = y v (z v x).
[para(21(a,1),32(a,1,2)),flip(a)].
104 0 v x = x. [para(85(a,1),9(a,1)),flip(a)].
149 c(x) v c(y v x) = c(x). [para(8(a,1),86(a,1,2,1,2))].
150 x v (c(y v c(x)) v z) = x v z. [para(86(a,1),10(a,1,1)),flip(a)].
164 c(x) v (y v c(x v z)) = y v c(x). [para(68(a,1),32(a,1,2)),flip(a)].
216 c(x v y) v c(y v c(x v y)) = c(y).
[para(149(a,1),21(a,1,2,1,2,1)),rewrite([8(6),9(5),149(11)])].
217 c(x) v (y v c(z v x)) = y v c(x). [para(149(a,1),32(a,1,2)),flip(a)].
456 x v c(x v c(y v c(x v y))) = 1.
[para(34(a,1),60(a,1,2,1,1)),rewrite([28(2),9(6),104(8)])].
787 x v c(y v c(y v c(x))) = x v c(y).
[para(73(a,1),150(a,1,2)),flip(a)].
1088 c(x v y) v (z v c(y v c(x v y))) = z v c(y).
[para(149(a,1),92(a,1,2,2,1,2,1)),rewrite([8(6),9(5),149(12)])].
3051 x v c(y v c(x v y)) = x.
[para(456(a,1),44(a,1,2,1)),rewrite([28(2),85(2),9(5)]),flip(a)].
3077 x v c(y v x) = x v c(y).
[para(3051(a,1),86(a,1,2,1)),rewrite([9(5),164(5)]),flip(a)].

```

```

3078  $x \vee c(x \vee c(y)) = y \vee x$ .
[para(3051(a,1),46(a,1,2,2,1,2,2,1,2)),rewrite([3077(3),3077(6),150(10),30
77(6),150(8),3077(7),86(8)])].
3239  $c(x \vee y) \vee (z \vee c(y \vee c(x))) = z \vee c(y)$ .
[back_rewrite(1088),rewrite([3077(5)])].
3285  $c(x \vee y) \vee c(y \vee c(x)) = c(y)$ .
[back_rewrite(216),rewrite([3077(5)])].
3299  $x \vee c(x \vee y) = x \vee c(y)$ . [back_rewrite(787),rewrite([3078(4)])].
3553  $x \vee (y \vee c(z \vee x)) = y \vee (x \vee c(z))$ .
[para(3077(a,1),32(a,1,2)),flip(a)].
3979  $c(x \vee c(y \vee z)) = c(x \vee c(z)) \vee c(x \vee (z \vee c(y)))$ .
[para(217(a,1),3285(a,1,1,1)),rewrite([8(8),9(7),3553(7)]),flip(a)].
4132 $F # answer("Foulis-Holland Theorem 4").
[back_rewrite(23),rewrite([3979(10),9(7),3979(8),9(5),9(13),10(13),9(15),9
(24),10(24),149(23),9(21),3239(21),8(11),9(10),3299(10)]),xx(a)].
===== end of proof =====

```

Figure 6. Summary of *prover9* proof of FH4.

4.0 Discussion and conclusions

The results of Section 3.0 motivate several observations:

1. *prover9* can derive the FHT from the axioms in Figure 1 conjoined with Df. 2.20.
2. The proofs in Section 3 use only *axioms* AxL1, AxL2, AxL3, AxL5, AxL6, AxOL1, AxOL2, AxOL3, and AxOM from Figure 1. None of the proofs use axiom AxL4.
3. Each of the proofs shown in Section 3 took less than four seconds to execute on the platform described in Section 2.0.

The derivations in Section 3.0 appear to be novel.

Using the *Mathematica* (Wolfram Research 2019) equational logic (McCune and Padmanabhan 1996) ADF “FindEquationalProof”, in an earlier version of this paper I attempted an automated proof of the FHT. Thanks in part to an astute observation by an anonymous referee of that earlier version, and by detailed analysis of the proof of the FHT contained in Megill 2019, I am now inclined to believe that a purely equational logic ADF approach to proving the FHT is likely to require much more explicit human control than the approach used in the present paper. Future work will investigate whether there is a relatively small set of definitions, which when conjoined with the axioms in Figure 1, can yield an “almost fully” automated proof of the FHT directly from the axioms of Figure 1.

5.0 Acknowledgements

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APPENDIX 1. prover9 input file for FH1.

formulas(assumptions).

% Lattice axioms

```
x = comp(comp(x))                # label("AxL1").
x v y = y v x                    # label("AxL2").
(x v y) v z = x v (y v z)       # label("AxL3").
(x ^ y) ^ z = x ^ (y ^ z)       # label("AxL4").
x v (x ^ y) = x                  # label("AxL5").
x ^ (x v y) = x                  # label("AxL6").
```

% Ortholattice axioms

```
comp(x) ^ x = 0                  # label("AxOL1").
comp(x) v x = 1                  # label("AxOL2").
x ^ y = comp(comp(x) v comp(y)) # label("AxOL3").
```

% Definition of "commutes"

```
C(x,y) <-> (x = ((x ^ y) v (x ^ comp(y)))) # label("Df: commutes").
```

% Other definitions

```
l_2 = (y v ((x ^ c(y)) v (c(x) ^ c(y)))) # label("Df. 2.20").
```

% Orthomodularity axiom (OMA)

```
y v (c(y) ^ (x v y)) = x v y # label("OMLaw").
```

end_of_list.

formulas(goals).

% Foulis-Holland theorem FH1

```
(C(x,y) & C(x,z)) -> ((x ^ (y v z)) = ((x ^ y) v (x ^ z)))
# label("Foulis-Holland Theorem 1").
```

end_of_list.

APPENDIX 2. prover9 input file for FH2.

formulas (assumptions).

```
% Lattice axioms
x = c(c(x))           # label("AxL1").
x v y = y v x         # label("AxL2").
(x v y) v z = x v (y v z) # label("AxL3").
(x ^ y) ^ z = x ^ (y ^ z) # label("AxL4").
x v (x ^ y) = x       # label("AxL5").
x ^ (x v y) = x       # label("AxL6").

% Ortholattice axioms
c(x) ^ x = 0         # label("AxOL1").
c(x) v x = 1         # label("AxOL2").
x ^ y = c(c(x) v c(y)) # label("AxOL3").

% Definition of x commutes with y
C(x,y) <-> (x = ((x ^ y) v (x ^ c(y)))) # label("Df: commutes").

% Other definitions
1_2 = (y v ((x ^ c(y)) v (c(x) ^ c(y)))) # label("Df. 2.20").

% Orthomodularity axiom (OMA)
y v (c(y) ^ (x v y)) = x v y # label("OMLaw").
```

end_of_list.

formulas (goals).

```
% Foulis-Holland theorem FH2
(C(x,y) & C(x,z)) -> ((y ^ (x v z)) = ((y ^ x) v (y ^ z)))
# label("Foulis-Holland Theorem 2").
```

end_of_list.

APPENDIX 3. prover9 input file for FH3.

formulas (assumptions).

```
% Lattice axioms
x = c(c(x)) # label("AxL1").
x v y = y v x # label("AxL2").
(x v y) v z = x v (y v z) # label("AxL3").
(x ^ y) ^ z = x ^ (y ^ z) # label("AxL4").
x v (x ^ y) = x # label("AxL5").
x ^ (x v y) = x # label("AxL6").

% Ortholattice axioms
c(x) ^ x = 0 # label("AxOL1").
c(x) v x = 1 # label("AxOL2").
x ^ y = c(c(x) v c(y)) # label("AxOL3").

% Definition of x commutes with y
C(x,y) <-> (x = ((x ^ y) v (x ^ c(y)))) # label("Df: commutes").

% Other definitions
1_2 = (y v ((x ^ c(y)) v (c(x) ^ c(y)))) # label("Df. 2.20").

% Orthomodularity axiom (OMA)
y v (c(y) ^ (x v y)) = x v y # label("OMLaw").
```

end_of_list.

formulas (goals).

```
% Foulis-Holland theorem FH3
(C(x,y) & C(x,z)) -> ((x v (y ^ z)) = ((x v y) ^ (x v z)))
# label("Foulis-Holland Theorem 3").
```

end_of_list.

APPENDIX 4. prover9 input file for FH4.

formulas(assumptions).

% Lattice axioms

```
x = c(c(x))           # label("AxL1").
x v y = y v x         # label("AxL2").
(x v y) v z = x v (y v z) # label("AxL3").
(x ^ y) ^ z = x ^ (y ^ z) # label("AxL4").
x v (x ^ y) = x       # label("AxL5").
x ^ (x v y) = x       # label("AxL6").
```

% Ortholattice axioms

```
c(x) ^ x = 0          # label("AxOL1").
c(x) v x = 1          # label("AxOL2").
x ^ y = c(c(x) v c(y)) # label("AxOL3").
```

% Definition of x commutes with y

```
C(x,y) <-> (x = ((x ^ y) v (x ^ c(y)))) # label("Df: commutes").
```

% Other definitions

```
1_2 = (y v ((x ^ c(y)) v (c(x) ^ c(y)))) # label("Df. 2.20").
```

% Orthomodularity axiom (OMA)

```
y v (c(y) ^ (x v y)) = x v y # label("OMLaw").
```

end_of_list.

formulas(goals).

% Foulis-Holland theorem FH4

```
(C(x,y) & C(x,z)) -> ((y v (x ^ z)) = ((y v x) ^ (y v z)))
# label("Foulis-Holland Theorem 4").
```

end_of_list.

6.0 References

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