A Conditionally Constructive Variant of Tarski's Elementary Geometry

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Abstract

Tarski's "weak-continuity" elementary geometry (TWCEG) is a first-order finite axiomatization of Euclidean geometry. TWCEG is consistent and incomplete. Its universal sentences are decidable. There is no known proof of the consistency or independence of TWCEG that uses a finite-domain model. Here I show that if we replace the "segment construction" axiom of TWCEG with a slightly weaker axiom, we obtain an elementary Euclidean geometry (which I call "CCWCEG") that is first-order, finitely axiomatized, incomplete, consistent, and independent. The proofs of the consistency and independence of CCWCEG require only finite-domain models. I further show that the universal sentences of CCWCEG are decidable.

Keywords: Tarski elementary geometry, automated deduction, consistency of Euclidean geometry

A conditionally constructive variant of Tarski's elementary geometry

1.0 Introduction

Tarski's "weak continuity" elementary[1] geometry (TWCEG), defined by Axioms A1 – A10 and Axiom CA, of Schwabhäuser, Szmielew, and Tarski 1983 (hereafter called "SST 1983"), is a first-order finite axiomatization of Euclidean geometry (Hilbert 1971; Heath 1925). TWCEG is consistent and incomplete (Tarski 1959). Its universal sentences are decidable (Tarski 1959). There is no known proof of the consistency or independence of TWCEG that uses a finite-domain model.

Figure 1 shows the axioms of TWCEG.

```
%----A1 - Reflexivity axiom for equidistance
     ∀xy(equidistant(x,y,y,x))

%----A2 - Transitivity axiom for equidistance
    ∀xyzvv''z(  ~ equidistant(x,y,z,v)
    | ~ equidistant(x,y,v'',w)
    | equidistant(z,v,v'',w) )


%----A3 Identity axiom for equidistance
    ∀xyz( ~ equidistant(x,y,z,z)
    | x = y )


%----A4 Segment construction axiom
    ∀xyuv ∃z(between(x,y,z) & equidistant(y,z,u,v))


%----A5 - Outer five-segment axiom
    ∀v'xyzx'y'z'(  ~ equidistant(x,y,x',y')
    | ~ equidistant(y,z,y',z')
    | ~ equidistant(x,v,x',v')
    | ~ equidistant(y,v,y',v')
    | ~ between(x,y,z)
    | ~ between(x',y',z')
    | x = y
    | equidistant(z,v,z',v') )

%----A6 - Identity axiom for betweenness
   ∀xy( ~ between(x,y,x)
```

[1] Tarski 1959 (p. 16) defines an "elementary" geometry as one which is formulated without appeal to any set-theoretic concepts.

```
    | x = y )

%----A7 - Inner Pasch axiom
    ∀txyzu ∃v(between(x,t,u) & between(y,u,z) -> between(x,v,y) &
           between(z,t,v))


%----A8 - Lower dimension axiom
    ∃xyz(~ between(x,y,z) & ~ between(y,z,x) & ~ between(z,x,y))


%----A9 - Upper dimension axiom
   ∀xyzvw( ~ equidistant(x,w,x,v)
    | ~ equidistant(y,w,y,v)
    | ~ equidistant(z,w,z,v)
    | between(x,y,z)
    | between(y,z,x)
    | between(z,x,y)
    | w = v )


%----A10 - Euclid's axiom
    ∀txyzu ∃vw(between(x,u,t) & between(y,u,z) & (x ≠ u) ->
            between(x,z,v) & between(x,y,w) & between(v,t,w) )


%----CA - Weakened continuity axiom
    ∀xyzx′z′u ∃y′(equidistant(u,x,u,x′) &
       equidistant(u,z,,u,z′) & between(u,x,z) & between(x,y,z) ->
       equidistant(u,y,u,y′) & between(x′,y′,z′) )
```

Figure 1. The axioms of TWCEG (SST 1983) in uninterpreted first-order-language form (see, for example, Church 1956). t, x, x′, y, y′ z, z′ u, v, v′, v′′, and w are first-order uninterpreted variables. *between* and *equidistant* are first-order uninterpreted relations. | ("or"), = ("equals"), ≠ ("not equal), & ("and"), -> ("implies"), and ~ ("not") are first-order logical connectives. Lines beginning with a percent sign (%) are comments.

In Figure 1, names (of variables and relations) are *uninterpreted* (strings of) symbols that satisfy the expression-formation rules of the language in which Figure 1 is cast. For example, "between(x,y,z)" does not *mean* that "the point y is strictly between points x and z", because *between* is an *un*interpreted name.

The TWCEG axioms, together with definitions cast in the vocabulary of those axioms, are sufficient to prove all of the theorems in Sections 1-15 of Part I of SST 1983. For the purpose of this paper, I will take the axioms of TWCEG, together with the theorems and definitions in those Sections, to be *a representation of elementary Euclidean geometry.*

One can prove the consistency of TWCEG by showing that a Cartesian space over a real closed field is a model (Chang and Keisler 2012, Section 1.3) of TWCEG (see in particular SST 1983, Satz 1.6). The domain of such a model has the cardinality of the reals. Any model of TWCEG is isomorphic to a Cartesian space over some Euclidean field (Tarski 1959, Theorem 6, p. 27).[2]

TWCEG is incomplete. *Proof.* There is at least one sentence (A11' of SST 1983) that satisfies the sentence-formation rules of TWCEG but is not provable in TWCEG. Q.E.D.

It is not known whether TWCEG is decidable.

The structure of the remainder of this paper is as follows. Section 2.0 describes a Euclidean geometry, "CCWCEG", created by replacing the "segment construction" axiom (A4) of TWCEG with a slightly weaker axiom. Section 3.0 proves the consistency of CCWCEG using a finite-domain model and Skolem theory. Section 4.0, together with the appendices of this paper, proves the independence of CCWCEG, using only finite-domain models. Section 5.0 proves that CCWCEG is incomplete. Section 6.0 discusses the decidability of CCWCEG. Section 7.0 contains acknowledgements. Section 8.0 contains the references for the paper. Sections 1.0 – 8.0 can be read standalone.

2.0 Conditional construction weak-continuity elementary geometry

TWCEG is first-order, finitely axiomatizable, incomplete, and consistent. Can we prove the consistency of TWCEG using a *finite*-domain model? All known attempts to date to find such a model have failed (see, for example, the "'GEO002*' TPTP Problems" in Sutcliffe and Suttner 2017). If we remove Axiom A4 from TWCEG, we obtain a theory whose consistency can proved by showing that a satisfaction-preserving transformation of that theory has a finite-domain model.[3] This consideration suggests that we might be able to weaken A4 of TWCEG slightly to obtain a system whose consistency can be proven (via Skolem theory; see Chang and Kiesler 2012, Section 3.3) with the use of a finite-domain model. Suppose we replace the connective "&" in A4 of TWCEG with "->", yielding

```
A4'(THE AXIOM OF CONDITIONAL SEGMENT CONSTRUCTION).
∀xyuv ∃z[between(x,y,z) -> equidistant(y,z,u,v)].
```

Figure 2. The axiom of conditional segment construction.

[2] The concepts of "Cartesian space" and "Euclidean field" do not presume the domain of the reals.
[3] #AUTHOR, unpublished results. More specifically, a Clause Normal Form (CNF, Sutcliffe 2016; TPTP Organization 2020) transformation of TWCEG without A4 has a finite model. By Skolem theory (Chang and Keisler 2012, Section 3.3), therefore, TWCEG without A4 has a model. See Section 3.0 of this paper for further detail on this proof technique.

I will call the axiom system resulting from replacing A4 in TWCEG with A4' of Figure 2, "conditionally constructive weak continuity elementary geometry" (CCWCEG).

Can we adopt CCWCEG as an alternative to TWCEG without fatally restricting the scope of what we mean by "a representation of elementary Euclidean geometry"? The answer is a qualified "yes". In particular, every theorem of TWCEG whose proof does not depend on A4 would remain a theorem under CCWCEG, because TWCEG and CCWCEG share all axioms except A4 of TWCEG. Now suppose we conjoin to the hypotheses of each of the theorems in TWCEG whose proofs require A4, the equivalent of the hypothesis of A4', i.e.,

$$\forall xy\ \exists z(\mathrm{between}(x,y,z))$$

Then the resulting theorems would be provable in CCWCEG.

The TWCEG theorems developed in SST 1983 that depend on A4 are shown in Table 1. Table 1 was derived by scanning Sections 1-15 of Part I of SST 1983 for occurrences of the string "A4", then analyzing those occurrences to determine whether they involve actual dependencies on A4. If a dependency was detected by this method, it was recorded in Table 1. The document was then scanned for all occurrences of the identifier (e.g. "3.6", "7.15") of any theorem in whose proof "A4" occurs, and those dependences were recorded in the table. This process was iterated until no further dependencies on "A4" were detected.

Theorem in SST 1983	**Page in SST 1983**
1.6	16
2.8	28
3.1	30
3.2	30 (Depends on 3.1)
3.3	30 (Depends on 3.1)
3.4	30 (Depends on 3.1)
3.5	30 (Depends on 3.1)
3.6	30 (Depends on 3.2 and 3.5)
3.7	31 (Depends on 3.6)
3.14	32
4.5	35
4.6	36 (Depends on 4.5)
4.14	37
6.10 (in a remark)	44
7.4	49
7.13	50
7.25	55
7.25 (existence theorem)	60
8.22	64-65
11.4	95
11.74	120

Table 1. Theorems in SST 1983 whose proofs depend on Axiom A4 of TWCEG.

These ~20 theorems constitute about 15% of the theorems explicitly stated in Sections 1-5 of Part I of SST 1983. The theorems in Table I, with their hypotheses augmented with the hypothesis of A4', together with the remaining theorems stated in Sections 1-15, thus constitute a representation of an elementary Euclidean geometry, as that term is defined above.

It might be objected that this method for determining which theorems in Sections 1-15 of SST 1983 depend on Axiom A4 of TWCEG could fail to capture all theorems that depend on that axiom. In particular, the proof of a theorem in those Sections might actually require A4, but the explication of that proof might have failed to mention A4. Or A4 might be cited in a proof in those Sections when the proof in fact does not depend on A4.

These concerns are less than they may seem for at least three reasons. First, a critical reading of SST 1983 reveals that its authors took great care to explicitly identify the dependencies of all proofs on A4 of TWCEG (and the dependencies on on any other axioms). Second, the items at the link mentioned in the reference for Beeson and Wos 2017 contain automated derivations of almost all of the theorems in Sections 1-9, and parts of Sections 10, 11, and 12, of Part I of SST 1983. By default, these derivations automatically document the axiom-dependence of those theorems, including all the theorems shown in Table 1 except 11.4 and 11.74. In any case, if a heretofore undocumented dependency of a theorem on A4 of TWCEG were to be discovered, we would simply add it to Table 1, and the hypotheses of the affected theorem could be augmented to depend on the hypotheses of A4' in the way noted above.

There is, in principle, an additional way in which A4' might be (implicitly) assumed by a proof in Sections 1-15 of SST 1983: A4' might not be independent of the other axioms of CCWCEG. Section 4.0, together with the Appendices) of this paper, show that CCWCEG is independent, and thus A4' is independent of all the other axioms of CCWCEG.

3.0 A finite-domain model-theoretic proof of "CCWCEG is consistent"

A *Clause Normal Form* (CNF; Sutcliffe 2016; TPTP Organization 2020) transformation of a set of first-order sentences S is a satisfaction- (but not implication-) preserving transformation of the members of S. Among other things, a CNF transformation eliminates existential quantifiers by Skolemizing the members of S (see Chang and Keisler 2012, Section 3.3). If a satisfaction-preserving transformation of S has a model M, then S is satisfiable, and therefore consistent (Chang and Keisler 2012, pp. 32-33). The task of this section is to produce a model of a CNF transformation of CCWCEG, which implies that CCWCEG is satisfiable (and therefore consistent).

A copy of a CNF transformation the axioms of TWCEG, expressed in the TPTP language (TPTP Organization 2020), was obtained from the TPTP Library (file GEO002-0.ax, Sutcliffe and Suttner 2017). The axioms for segment_construction_1 and segment_construction_2 in file GEO002-0.ax were replaced with a CNF transformation of Axiom A4' ("conditional_segment_construction"). A listing of the resulting CNF/TPTP-formatted file, GEO002-0.cond_seg.ax, is shown in Figure 3.

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Identity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


%----A4' Conditional segment construction axiom
cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
```

```
    | between(X,inner_pasch(U,V,W,X,Y),U) )).

%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension_point_3
) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension_point_1
) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension_point_2
) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.[4]
```

[4] With the exception of this ("weakened continuity") axiom, the identifiers (e.g., "A5", "A9", etc.) of the axioms in Figure 3 correspond exactly to the axiom-identifiers in SST 1983. The axiom-identifier "A11" in Figure 3 can be a source of confusion. The CNF clause corresponding to this "All" comment shown here and in subsequent variants of CCWCEG in this paper is called "CA" ("circle axiom"; in SST 1983,

```
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------------
```

Figure 3. Listing of file GEO002-0.cond_seg.ax, a CNF transformation of Figure 1 in which the CNF transformation of Axiom A4 has been replaced by the CNF transformation of A4'. All expressions that begin with a capital letter (e.g., X, Z1) are Prolog-style variables (see Ed-Deali, Deransart, and Cervoini 1996). Lines beginning with the percent sign (%) are comments.

Here is some guidance on how to read the axioms in Figure 3. Consider in particular

```
cnf(inner_pasch1,axiom,
  ( ~ between(U,V,W)
  | ~ between(Y,X,W)
  | between(V,inner_pasch(U,V,W,X,Y),Y) )).
```

This expression is of the form

```
cnf(expression_name, type_of_expression,
    ( expression_in_cnf ) )
```

In the example, "cnf" denotes that the expression between the outermost set of parentheses is in CNF form. "outer_pasch1" is the name of the expression realized in *expression_in_cnf*. "axiom" denotes that the *type* of the *expression_in_cnf* is *axiom*. The expressions in *expression_in_cnf* are delimited by "|" ("or"). In the jargon of CNF, any realization of *expression_in_cnf* is called a *clause*. The third of these three expressions in the clause for "inner_pasch1",

```
between(V,inner_pasch(U,V,W,X,Y),Y)
```

"Kreisaxiom") in SST 1983. Axiom "A11" in SST 1983, in contrast, denotes a stronger, *second*-order continuity axiom. [The axiom-identifier "A11" mentioned in Figure 3 comes from the TPTP file GEO002-0.ax (Sutcliffe and Suttner 2017). I have chosen to retain that identifier for the purposes of traceability into the TPTP Library (Sutcliffe and Suttner 2017).]

contains a Skolem function

```
inner_pasch(U,V,W,X,Y) .
```

For further detail on TPTP syntax, see TPTP Organization 2020. For a tutorial on CNF, see Sutcliffe 2016.

The automated first-order derivation and finite-model generating software system VAMPIRE (Kovács and Voronkov 2013) was used to generate a finite model of the system shown in Figure 3. In particular, under Red Hat Fedora (a Linux variant) running on a Dell Inspiron 3650 (containing an Intel i5-6400 quadprocessor clocked at 2.6 GHz, 8 GB physical memory, and 1 TB disk), VAMPIRE was invoked from the command line by

```
$ ./vampire_z3_Release_static_master_4764
  --saturation_algorithm fmb GEO002-0.cond_seg.ax >
          GEO002-0.cond_seg.fmb.txt
```

where "$" is the command-line prompt. The output of this command, **GEO002-0.cond_seg.fmb.txt**, which includes a model for the system shown in Figure 3, is shown in Figure 4.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg
% SZS output start FiniteModel for GEO002-0.cond_seg

tff(declare_$i,type,$i:$tType).

tff(declare_$i1,type,lower_dimension_point_1:$i).

tff(finite_domain,axiom,! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).

tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).

tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).
```

```
tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,
equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,
~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)).

% SZS output end FiniteModel for GEO002-0.cond_seg
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.001 s
% ------------------------------
% ------------------------------
```

Figure 4. A finite-domain model of Figure 3, generated by VAMPIRE.

Here is some guidance on how to read Figure 4. The first four, and last eight, non-blank lines in Figure 4 contain information about (a) whether VAMPIRE found a model and (b), some configuration and resource-usage for the particular invocation of VAMPIRE shown in Figure 4.

Figure 4 assigns values to a subset of the expressions in Figure 3. For example,

```
tff(finite_domain,axiom,! [X:$i] : (
          X = lower_dimension_point_1
      ) ).
```

assigns the value "lower_dimension_point_1" to all the variables in Figure 3. Similarly,

```
tff(function_extension,axiom,
extension(lower_dimension_point_1,lower_dimension_point_1,
lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1).
```

assigns the value "lower_dimension_point_1" to the Skolem function "extension" when the values of all the arguments of "extension" have value "lower_dimension_point_1". For further information on how to read Figure 4, see TPTP Organization 2020.

Given the value-assignments contained in Figure 4, the axioms in Figure 3 are satisfied, i.e., Figure 4 contains a finite-domain model of the axiom set in Figure 3.[5] By Skolem theory, CCWCEG (Chang and Keisler 2012, Section 3.3) is satisfiable.[6] Q.E.D.

Note that unlike the traditional proofs of the consistency of TWCEG, the model of the CNF transformation of CCWCEG shown in Figure 4 nowhere appeals to the concepts of a real closed field, a Euclidean field, or a Cartesian space.

4.0 Independence of CCWCEG

Section 2 raises the possibility that A4' might not be independent of the remaining axioms of CCWCEG. If that were so, A4' could be removed from CCWCEG without loss of implicational scope.

If we replace the CNF transformation of a given axiom, A, of CCWCEG that occurs in Figure 3 with the CNF transformations of the negation of A, then use the command-line noted in Section 3 as a template for invoking VAMPIRE, we obtain a model showing the independence of A. If we perform this operation for each axiom in CCWCEG (one axiom at a time), we obtain the models shown in the Appendices. It follows by Skolem theory that the axioms of CCWCEG are independent.

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[5] This claim can be verified by evaluating Figure 3 under the value-assignments of Figure 4. Note that Figure 4 is in one sense rather degenerate: for every expression to which it assigns a value, it assigns the same value (`lower_dimension_point_1`). That degeneracy makes it relatively to show that Figure 4 satisfies Axioms A3, A5, A6, A9, A10.1, A10.2, and A10.3 in Figure 3 because they each contain a disjunct of the form "$\alpha = \beta$", where α and β are *metavariables* ranging over all variables in Figure 3.

[6] In Skolem theory, the existence of a finite model of the CNF transformation of a theory T implies that there is *a* model of T, but does not imply that there is a *finite* model of T.

Again, note that unlike the traditional proofs of the consistency of TWCEG, the model shown in the Appendices nowhere appeals to the concepts of a real closed field, a Euclidean field, or a Cartesian space.

VAMPIRE generated just six different models for the proofs of the independence of the CCWCEG axioms. In particular, the models for the proofs of the independence of each of Axioms A1, A2, A3, A5, A7.1, A7.2, A9, A10.1, A10.2, A10.3, 11.1, and 11.3 are the same. The models for the proofs of the independence of each of Axiom A4' and Axiom A6 are each unique. The models used in the proofs of the independence of each of Axioms A8.1, A8.2, and A8.3 differ, but are highly symmetric.

5.0 Completeness

CCWCEG is incomplete. *Proof.* TWCEG and CCWCEG share the same sentence-formation rules. A4 of TWCEG implies A4' of CCWCEG, but not conversely. Thus, there is at least one sentence (A4) satisfying the sentence-formation rules of CCWCEG that is not derivable from CCWCEG. Q.E.D.

6.0 Decidability

The universal sentences[7] of CCWCEG are decidable. *Proof.* The universal sentences of CCWCEG are the same as the universal sentences of Tarski's elementary geometry E2'' (Tarski 1959, p. 20). The universal sentences of E2'' are decidable (Tarski 1959, Theorem 8, p. 28). Thus, the universal sentences of CCWCEG are decidable. Q.E.D.

Whether all the non-universal sentences of CCWCEG are decidable is unknown.

7.0 Discussion and conclusions

Sections 1-6 of this paper motivate several observations:

1. Any theorem of TWCEG that does not depend on the "segment construction" Axiom A4 of TWCEG is derivable from the axioms of CCWCEG, because with the exception of A4, the axioms of CCWCEG are those of TWCEG. Any theorem R of TWCEG that depends on A4 can be transformed to a theorem of CCWCEG by conjoining to the hypotheses of R the hypothesis of the "conditionally constructive" axiom A4'. Thus if we replace A4 of TWCEG with A4' and modify, as noted in Section 2.0, the theorems of TWCEG that depend on A4, we obtain an elementary Euclidean geometry, CCWCEG, that is first-order, finitely axiomatized, consistent, independent, and incomplete. Section 3.0 proves

[7] For a definition of "universal sentence, see Tarski 1959, p. 24.

the consistency of CCWCEG (via Skolem theory) using only finite-domain model; Section 4.0 proves the independence of CCWCEG (again, via Skolem theory), using only finite-domain models. Section 5.0 shows that CCWCEG is incomplete, and Section 6.0 shows that all the universal sentences of CCWCEG are decidable.

2. Unlike the traditional proofs of the consistency of TWCEG, the proof of the consistency of CCWCEG shown in Figure 4, and the proof of the independence of CCWCEG contained in the Appendices, nowhere appeal to the concepts of a real closed field, a Euclidean field, or a Cartesian space.[8]
3. The model-groupings mentioned in Section 4.0 suggest that the axiom-sets corresponding to those groups may represent interesting natural subtheories of CCWCEG. Future work will address this conjecture.

8.0 Acknowledgements

This work benefited from discussions with John Symons. I learned much about the art of automated deduction from Art Quaife's (1992) Otter-based (McCune 2003) automated deduction framework for TWCEG, and from the Stanford computational metaphysics project (Zalta et al. 2020). I am also indebted to Arthur Skidmore and Alberto Coffa, whose passion for formal methods was an inspiration to all who were privileged to have known them.

[8] Thanks to John Symons for this observation .

APPENDICES

The Appendices contain models that show the independence of each of the axioms of CCWCEG.

Each of the appendices contains two sections: an input-script listing, and a corresponding model-file listing. Each model listing was generated by VAMPIRE by the command

```
$ ./vampire_z3_Release_static_master_4764
     --saturation_algorithm fmb input_script >
       model_file
```

where *input_script* is the name of the file containing the script for which a model is to be generated, and *model_file* is the name of the file containing the model for the axiom set in *input_script*. **vampire_z3_Release_static_master_4764** is the name of the VAMPIRE executable (see link at the reference for Kovács and Voronkov 2013) used for these Appendices.

Any input script shown in these Appendices differ from the other input scripts in the Appendices only by the CNF transformation of the negation of the single axiom in Figure 3 whose independence is to be proven by the script.

All models were generated by executing instances of the command above, running under Red Hat Fedora (a Linux variant) on a Dell Inspiron 3650 (containing an Intel i5-6400 quadprocessor clocked at 2.6 GHz, 8 GB physical memory, and 1 TB disk).

APPENDIX 1. Negate A1

Negate A1 input script

```
%----A1 - Negate Reflexivity axiom for equidistance
cnf(neg_reflexivity_for_equidistance,axiom,
    ( ~  equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).
```

Appendix 1, continued

Model produced by VAMPIRE for Negate A1.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA1
% SZS output start FiniteModel for GEO002-0.cond_seg.negA1
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).
```

```
tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA1
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 2. Negate A2.

Negate A2 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Negate Transitivity axiom for equidistance
cnf(neg_transitivity_for_equidistance,axiom,
    (  equidistant(X,Y,Z,V)
    |  equidistant(X,Y,V2,W)
    |  ~ equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
```

```
    | between(X,inner_pasch(U,V,W,X,Y),U) )).

%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
```

```
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%--------------------------------------------------------------------------
----
```

Appendix 2, continued

Model produced by VAMPIRE for Negate A2.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA2
% SZS output start FiniteModel for GEO002-0.cond_seg.negA2
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA2
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 3. Negate A3.

Negate A3 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Negate Indentity axiom for equidistance
cnf(neg_identity_for_equidistance,axiom,
    (  equidistant(X,Y,Z,Z)
    | ~ (X = Y) )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).
```

Appendix 3, continued

Model produced by VAMPIRE for Negate A3.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA3
% SZS output start FiniteModel for GEO002-0.cond_seg.negA3
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA3
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
```

APPENDIX 4. Negate A4'

Negate A4' input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).

%----A4' Negate Conditional segment construction
cnf(neg_conditional_segment_construction,axiom,
    (  between(X,Y,extension(X,Y,W,V))
    | ~ equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).
```

Appendix 4, continued

Model produced by VAMPIRE for Negate A4'.

```
TRYING [1]
TRYING [2]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA4
% SZS output start FiniteModel for GEO002-0.cond_seg.negA4
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(declare_$i2,type,fmb_$i_2:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1 | X = fmb_$i_2
      ) ).

tff(distinct_domain,axiom,
         lower_dimension_point_1 != fmb_$i_2
).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = fmb_$i_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,fmb_$i_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lowe
r_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb_
$i_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lowe
r_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb_
$i_2) = fmb_$i_2
         &
extension(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_poi
nt_1) = fmb_$i_2
         &
extension(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1
```

```
        &
extension(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lowe
r_dimension_point_1) = fmb_$i_2
        &
extension(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb_
$i_2) = lower_dimension_point_1
        &
extension(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_poi
nt_1) = lower_dimension_point_1
        &
extension(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
        &
extension(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_poi
nt_1) = lower_dimension_point_1
        &
extension(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2) =
fmb_$i_2
        &
extension(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1) =
fmb_$i_2
        & extension(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lo
wer_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lo
wer_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fm
b_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fm
b_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lo
wer_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
```

```
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lo
wer_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fm
b_$i_2,lower_dimension_point_1) = lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fm
b_$i_2,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_p
oint_1,lower_dimension_point_1) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_p
oint_1,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_di
mension_point_1) = fmb_$i_2
        &
inner_pasch(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2
) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lo
wer_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lo
wer_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fm
b_$i_2,lower_dimension_point_1) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fm
b_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_p
oint_1,lower_dimension_point_1) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_p
oint_1,fmb_$i_2) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_di
mension_point_1) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2
) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_p
oint_1,lower_dimension_point_1) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_p
oint_1,fmb_$i_2) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_di
mension_point_1) = fmb_$i_2
```

```
        &
inner_pasch(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2
) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_di
mension_point_1) = lower_dimension_point_1
        &
inner_pasch(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2
) = fmb_$i_2
        &
inner_pasch(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1
) = fmb_$i_2
        & inner_pasch(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,fmb_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lower_
dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lower_
dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb_$i
_2,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb_$i
_2,fmb_$i_2) = lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lower_
dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lower_
dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb_$i
_2,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb_$i
_2,fmb_$i_2) = lower_dimension_point_1
```

```
        &
euclid1(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_point
_1,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_point
_1,fmb_$i_2) = lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimens
ion_point_1) = fmb_$i_2
        &
euclid1(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
        &
euclid1(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lower_
dimension_point_1,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid1(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lower_
dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
euclid1(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb_$i
_2,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid1(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb_$i
_2,fmb_$i_2) = fmb_$i_2
        &
euclid1(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_point
_1,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_point
_1,fmb_$i_2) = fmb_$i_2
        &
euclid1(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimens
ion_point_1) = lower_dimension_point_1
        &
euclid1(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_point
_1,lower_dimension_point_1) = fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_point
_1,fmb_$i_2) = fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimens
ion_point_1) = fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1
        &
euclid1(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimens
ion_point_1) = fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2) =
fmb_$i_2
        &
euclid1(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1) =
lower_dimension_point_1
```

```
          & euclid1(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lower_
dimension_point_1,lower_dimension_point_1) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,lower_
dimension_point_1,fmb_$i_2) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb_$i
_2,lower_dimension_point_1) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb_$i
_2,fmb_$i_2) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lower_
dimension_point_1,lower_dimension_point_1) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,lower_
dimension_point_1,fmb_$i_2) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb_$i
_2,lower_dimension_point_1) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb_$i
_2,fmb_$i_2) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_point
_1,lower_dimension_point_1) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_point
_1,fmb_$i_2) = lower_dimension_point_1
          &
euclid2(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimens
ion_point_1) = fmb_$i_2
          &
euclid2(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
```

```
        &
euclid2(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lower_
dimension_point_1,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,lower_
dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
euclid2(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb_$i
_2,lower_dimension_point_1) = fmb_$i_2
        &
euclid2(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb_$i
_2,fmb_$i_2) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_point
_1,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_point
_1,fmb_$i_2) = fmb_$i_2
        &
euclid2(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimens
ion_point_1) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
        &
euclid2(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_point
_1,lower_dimension_point_1) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_point
_1,fmb_$i_2) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimens
ion_point_1) = lower_dimension_point_1
        &
euclid2(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2) =
fmb_$i_2
        &
euclid2(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimens
ion_point_1) = fmb_$i_2
        &
euclid2(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2) =
lower_dimension_point_1
        &
euclid2(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1) =
lower_dimension_point_1
        & euclid2(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1

).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2) =
fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1) =
fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,fmb_$i_2,fmb_$i_2) =
lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,fmb_$i_2,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,low
er_dimension_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,low
er_dimension_point_1,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,low
er_dimension_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,low
er_dimension_point_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb
_$i_2,lower_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb
_$i_2,lower_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb
_$i_2,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,fmb
_$i_2,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,low
er_dimension_point_1,lower_dimension_point_1,lower_dimension_point_1) =
fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,low
er_dimension_point_1,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
```

```
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,low
er_dimension_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,low
er_dimension_point_1,fmb_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb
_$i_2,lower_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb
_$i_2,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb
_$i_2,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fmb
_$i_2,fmb_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_po
int_1,lower_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_po
int_1,lower_dimension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_po
int_1,fmb_$i_2,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_po
int_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dim
ension_point_1,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dim
ension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,
lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,
fmb_$i_2) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,low
er_dimension_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,low
er_dimension_point_1,lower_dimension_point_1,fmb_$i_2) =
lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,low
er_dimension_point_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,low
er_dimension_point_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
```

```
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb
_$i_2,lower_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb
_$i_2,lower_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb
_$i_2,fmb_$i_2,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fmb
_$i_2,fmb_$i_2,fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_po
int_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_po
int_1,lower_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_po
int_1,fmb_$i_2,lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_po
int_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dim
ension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dim
ension_point_1,fmb_$i_2) = fmb_$i_2
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,
lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2,
fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_po
int_1,lower_dimension_point_1,lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_po
int_1,lower_dimension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_po
int_1,fmb_$i_2,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_po
int_1,fmb_$i_2,fmb_$i_2) = fmb_$i_2
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dim
ension_point_1,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dim
ension_point_1,fmb_$i_2) = fmb_$i_2
```

```
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,
lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2,
fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dim
ension_point_1,lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dim
ension_point_1,fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,
lower_dimension_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2,
fmb_$i_2) = fmb_$i_2
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,
lower_dimension_point_1) = fmb_$i_2
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1,
fmb_$i_2) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension
_point_1) = lower_dimension_point_1
        &
continuous(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2) =
fmb_$i_2

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,fmb_$i_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,l
ower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2,f
mb_$i_2)
        &
~equidistant(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,l
ower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1,fm
b_$i_2)
        &
equidistant(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,lower_dimension_p
oint_1)
```

```
          &
~equidistant(lower_dimension_point_1,fmb_$i_2,fmb_$i_2,fmb_$i_2)
          &
~equidistant(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,l
ower_dimension_point_1)
          &
equidistant(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1,fm
b_$i_2)
          &
equidistant(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,lower_dimension_p
oint_1)
          &
~equidistant(fmb_$i_2,lower_dimension_point_1,fmb_$i_2,fmb_$i_2)
          &
~equidistant(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,lower_dimension_
point_1)
          &
~equidistant(fmb_$i_2,fmb_$i_2,lower_dimension_point_1,fmb_$i_2)
          &
~equidistant(fmb_$i_2,fmb_$i_2,fmb_$i_2,lower_dimension_point_1)
          & equidistant(fmb_$i_2,fmb_$i_2,fmb_$i_2,fmb_$i_2)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)
          &
~between(lower_dimension_point_1,lower_dimension_point_1,fmb_$i_2)
          &
~between(lower_dimension_point_1,fmb_$i_2,lower_dimension_point_1)
          & ~between(lower_dimension_point_1,fmb_$i_2,fmb_$i_2)
          &
~between(fmb_$i_2,lower_dimension_point_1,lower_dimension_point_1)
          & ~between(fmb_$i_2,lower_dimension_point_1,fmb_$i_2)
          & ~between(fmb_$i_2,fmb_$i_2,lower_dimension_point_1)
          & between(fmb_$i_2,fmb_$i_2,fmb_$i_2)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA4
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 5373
% Time elapsed: 0.003 s
% ------------------------------
% ------------------------------
```

APPENDIX 5. Negate A5.

Negate A5 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Negate Outer five-segment axiom
cnf(neg_outer_five_segment,axiom,
    (  equidistant(X,Y,X1,Y1)
    |  equidistant(Y,Z,Y1,Z1)
    |  equidistant(X,V,X1,V1)
    |  equidistant(Y,V,Y1,V1)
    |  between(X,Y,Z)
    |  between(X1,Y1,Z1)
    | ~ (X = Y)
    | ~  equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 5, continued

Model produced by VAMPIRE for Negate A5.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA5
% SZS output start FiniteModel for GEO002-0.cond_seg.negA5
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA5
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 6. Negate A6.

Negate A6 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Negate Identity axiom for betweenness
cnf(neg_identity_for_betweeness,axiom,
    (  between(X,Y,X)
    | ~ (X = Y) )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------------
----
```

Appendix 6, continued

Model produced by VAMPIRE for Negate A6.

```
TRYING [1]
TRYING [2]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA6
% SZS output start FiniteModel for GEO002-0.cond_seg.negA6
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(declare_$i2,type,lower_dimension_point_2:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1 | X = lower_dimension_point_2
      ) ).

tff(distinct_domain,axiom,
         lower_dimension_point_1 != lower_dimension_point_2
).

tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2
         &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
```

```
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2

).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_2)
```

```
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

between(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_2,lower_dimension_point_2,lower_dimensio
n_point_1)
        &
between(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA6
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 5500
% Time elapsed: 0.003 s
% ------------------------------
% ------------------------------
```

APPENDIX 7. Negate A7.1

Negate A7.1 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 -Negate  Inner Pasch axiom, two clauses.
%---- Negate A7.1
cnf(neg_inner_pasch1,axiom,
    (  between(U,V,W)
    |  between(Y,X,W)
    | ~ between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 7, continued

Model produced by VAMPIRE for Negate A7.1.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA7.1
% SZS output start FiniteModel for GEO002-0.cond_seg.negA7.1
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA7.1
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 8. Negate A7.2

Negate A7.2 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Negate Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%---- Negate 7.2
cnf(neg_inner_pasch2,axiom,
    (  between(U,V,W)
    |  between(Y,X,W)
    | ~ between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%--------------------------------------------------------------------------
----
```

Appendix 8, continued

Model produced by VAMPIRE for Negate A7.2.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA7.2
% SZS output start FiniteModel for GEO002-0.cond_seg.negA7.2
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA7.2
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.001 s
% ------------------------------
% ------------------------------
```

APPENDIX 9. Negation of A8.1

Negate A8.1 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Negate Lower dimension axiom, three clauses.
%---- negate A8.1
cnf(neg_lower_dimension1,axiom,
    (
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 9, continued

Model produced by VAMPIRE for Negate A8.2.

```
TRYING [1]
TRYING [2]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA8.1
% SZS output start FiniteModel for GEO002-0.cond_seg.negA8.1
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(declare_$i2,type,lower_dimension_point_3:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1 | X = lower_dimension_point_3
      ) ).

tff(distinct_domain,axiom,
         lower_dimension_point_1 != lower_dimension_point_3
).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_3) = lower_dimension_point_3
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_3,lower_dimension_point_1) = lower_dimension_point_3
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_3,lower_dimension_point_3) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_3,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_3,lower_dimensi
on_point_1,lower_dimension_point_3) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_3,lower_dimensi
on_point_3,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_3,lower_dimensi
on_point_3,lower_dimension_point_3) = lower_dimension_point_1
         &
extension(lower_dimension_point_3,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_3
```

```
        &
extension(lower_dimension_point_3,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_3) = lower_dimension_point_3
        &
extension(lower_dimension_point_3,lower_dimension_point_1,lower_dimensi
on_point_3,lower_dimension_point_1) = lower_dimension_point_3
        &
extension(lower_dimension_point_3,lower_dimension_point_1,lower_dimensi
on_point_3,lower_dimension_point_3) = lower_dimension_point_3
        &
extension(lower_dimension_point_3,lower_dimension_point_3,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_3
        &
extension(lower_dimension_point_3,lower_dimension_point_3,lower_dimensi
on_point_1,lower_dimension_point_3) = lower_dimension_point_1
        &
extension(lower_dimension_point_3,lower_dimension_point_3,lower_dimensi
on_point_3,lower_dimension_point_1) = lower_dimension_point_1
        &
extension(lower_dimension_point_3,lower_dimension_point_3,lower_dimensi
on_point_3,lower_dimension_point_3) = lower_dimension_point_3

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
```

```
sion_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
```

```
sion_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
inner_pasch(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
```

```
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
```

```
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
         &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
         &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_1,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
         &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_1,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_3
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_1,lower_dimension_point_3) =
lower_dimension_point_3
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_3,lower_dimension_point_3,lower_dimension
_point_3,lower_dimension_point_3,lower_dimension_point_3) =
lower_dimension_point_1

).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_1,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3
```

```
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_1,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_3
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3) = lower_dimension_point_3
```

```
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_3,lower_dimension_point_3,lower_dimens
ion_point_3,lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3) = lower_dimension_point_3

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_3)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_3,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_3)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_3,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_3)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_3,lower_dime
nsion_point_3,lower_dimension_point_3)
        &
~equidistant(lower_dimension_point_3,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_3)
        &
equidistant(lower_dimension_point_3,lower_dimension_point_1,lower_dimen
sion_point_3,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_3,lower_dimension_point_1,lower_dime
nsion_point_3,lower_dimension_point_3)
        &
equidistant(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_3,lower_dimension_point_3,lower_dime
nsion_point_1,lower_dimension_point_3)
```

```
        &
~equidistant(lower_dimension_point_3,lower_dimension_point_3,lower_dime
nsion_point_3,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_3,lower_dimension_point_3,lower_dimen
sion_point_3,lower_dimension_point_3)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

between(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1)
        &
between(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_3)
        &
~between(lower_dimension_point_1,lower_dimension_point_3,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_3,lower_dimensio
n_point_3)
        &
~between(lower_dimension_point_3,lower_dimension_point_1,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_3,lower_dimension_point_1,lower_dimensio
n_point_3)
        &
~between(lower_dimension_point_3,lower_dimension_point_3,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_3,lower_dimension_point_3,lower_dimensio
n_point_3)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA8.1
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 5500
% Time elapsed: 0.003 s
% ------------------------------
% ------------------------------
```

APPENDIX 10. Negate A8.2

Negate A8.2 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Negate Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%--- Negate A8.2
cnf(neg_lower_dimension2,axiom,
    (
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 10 , continued

Model produced by VAMPIRE for Negate A8.2.

```
TRYING [1]
TRYING [2]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA8.2
% SZS output start FiniteModel for GEO002-0.cond_seg.negA8.2
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(declare_$i2,type,lower_dimension_point_2:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1 | X = lower_dimension_point_2
      ) ).

tff(distinct_domain,axiom,
         lower_dimension_point_1 != lower_dimension_point_2
).

tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_2).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
```

```
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1

).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_2)
```

```
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_2)
        &
between(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1)
        &
between(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA8.2
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 5500
% Time elapsed: 0.003 s
% ------------------------------
% ------------------------------
```

APPENDIX 11. Negate A8.3

Negate A8.3 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Negate Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----negate A8.3
cnf(neg_lower_dimension3,axiom,
    (
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 11, continued

Model produced by VAMPIRE for Negate A8.3.

```
TRYING [1]
TRYING [2]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA8.3
% SZS output start FiniteModel for GEO002-0.cond_seg.negA8.3
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(declare_$i2,type,lower_dimension_point_2:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1 | X = lower_dimension_point_2
      ) ).

tff(distinct_domain,axiom,
         lower_dimension_point_1 != lower_dimension_point_2
).

tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
         &
extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
         &
extension(lower_dimension_point_1,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_1
         &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
```

```
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_1,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_2
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_1,lower_dimension_point_2) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_1) = lower_dimension_point_1
        &
extension(lower_dimension_point_2,lower_dimension_point_2,lower_dimensi
on_point_2,lower_dimension_point_2) = lower_dimension_point_2

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
```

```
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
inner_pasch(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid1(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_1,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
```

```
_point_1,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_1,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_1,lower_dimension_point_2) =
lower_dimension_point_2
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_1) =
lower_dimension_point_1
        &
euclid2(lower_dimension_point_2,lower_dimension_point_2,lower_dimension
_point_2,lower_dimension_point_2,lower_dimension_point_2) =
lower_dimension_point_1

).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_1,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_1,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_1,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_2
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2) = lower_dimension_point_2
```

```
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1) = lower_dimension_point_1
        &
continuous(lower_dimension_point_2,lower_dimension_point_2,lower_dimens
ion_point_2,lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2) = lower_dimension_point_2

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_1,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_1,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dimen
sion_point_2,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_1,lower_dime
nsion_point_2,lower_dimension_point_2)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_1,lower_dimension_point_1)
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_1,lower_dimension_point_2)
```

```
        &
~equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dime
nsion_point_2,lower_dimension_point_1)
        &
equidistant(lower_dimension_point_2,lower_dimension_point_2,lower_dimen
sion_point_2,lower_dimension_point_2)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

between(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1)
        &
between(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_2)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_1,lower_dimension_point_2,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_2,lower_dimension_point_1,lower_dimensio
n_point_2)
        &
~between(lower_dimension_point_2,lower_dimension_point_2,lower_dimensio
n_point_1)
        &
~between(lower_dimension_point_2,lower_dimension_point_2,lower_dimensio
n_point_2)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA8.3
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 5500
% Time elapsed: 0.003 s
% ------------------------------
% ------------------------------
```

APPENDIX 12. Negate A9.

Negate A9 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Negate Upper dimension axiom
cnf(neg_upper_dimension,axiom,
    (  equidistant(X,W,X,V)
    |  equidistant(Y,W,Y,V)
    |  equidistant(Z,W,Z,V)
    | ~ between(X,Y,Z)
    | ~ between(Y,Z,X)
    | ~ between(Z,X,Y)
    | ~( W = V) )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------------
----
```

Appendix 12, continued

Model produced by VAMPIRE for Negate A9.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA9
% SZS output start FiniteModel for GEO002-0.cond_seg.negA9
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).
```

```
tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA9
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.001 s
% ------------------------------
% ------------------------------
```

APPENDIX 13. Negate A10.1

Negate A10.1 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Negate Euclid's axiom, three clauses.
%----Negate A10.1
cnf(neg_euclid1,axiom,
    (  between(U,W,Y)
    |  between(V,W,X)
    | ~ (U = W)
    | ~ between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
```

```
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------------
----
```

Appendix 13, continued

Model produced by VAMPIRE for Negate A10.1.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA10.1
% SZS output start FiniteModel for GEO002-0.cond_seg.negA10.1
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).
```

```
tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA10.1
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 14. Negation of A10.2.

Negate A10.2 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Negate Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----negate A10.2
cnf(neg_euclid2,axiom,
    (  between(U,W,Y)
    |  between(V,W,X)
    | ~ (U = W)
    | ~  between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
```

```
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%--------------------------------------------------------------------------
----
```

Appendix 14, continued

Model produced by VAMPIRE for Negate A10.2.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA10.2
% SZS output start FiniteModel for GEO002-0.cond_seg.negA10.2
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).
```

```
tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA10.2
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 15. Negation of 10.3.

Negate A10.3 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
```

```
    | between(X,inner_pasch(U,V,W,X,Y),U) )).

%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Negate Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%---- negate A10.3
cnf(neg_euclid3,axiom,
    (  between(U,W,Y)
    |  between(V,W,X)
    | ~ (U = W)
    | ~  between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
```

```
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%--------------------------------------------------------------------------
----
```

Appendix 15, continued

Model produced by VAMPIRE for Negate A10.3.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA10.3
% SZS output start FiniteModel for GEO002-0.cond_seg.negA10.3
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).
```

```
tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA10.3
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.001 s
% ------------------------------
% ------------------------------
```

APPENDIX 16. Negate A11.1.

Negate A11.1 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(X,inner_pasch(U,V,W,X,Y),U) )).
```

```
%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Negate Weakened continuity axiom, two clauses.
%---- negate A11.1
cnf(neg_continuity1,axiom,
    (  equidistant(U,V,U,V1)
    |  equidistant(U,X,U,X1)
    |  between(U,V,X)
```

```
    |  between(V,W,X)
    | ~ between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%----A11.2
cnf(continuity2,axiom,
    ( ~ equidistant(U,V,U,V1)
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%----------------------------------------------------------------------
----
```

Appendix 16, continued

Model produced by VAMPIRE for Negate A11.1.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA11.1
% SZS output start FiniteModel for GEO002-0.cond_seg.negA11.1
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA11.1
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

APPENDIX 17. Negate A11.2

Negate A11.2 input script

```
%----A1 - Reflexivity axiom for equidistance
cnf(reflexivity_for_equidistance,axiom,
    ( equidistant(X,Y,Y,X) )).

%----A2 - Transitivity axiom for equidistance
cnf(transitivity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,V)
    | ~ equidistant(X,Y,V2,W)
    | equidistant(Z,V,V2,W) )).

%----A3 Indentity axiom for equidistance
cnf(identity_for_equidistance,axiom,
    ( ~ equidistant(X,Y,Z,Z)
    | X = Y )).


cnf(conditional_segment_construction,axiom,
    ( ~ between(X,Y,extension(X,Y,W,V))
    | equidistant(Y,extension(X,Y,W,V),W,V) )).


%----A5 - Outer five-segment axiom
cnf(outer_five_segment,axiom,
    ( ~ equidistant(X,Y,X1,Y1)
    | ~ equidistant(Y,Z,Y1,Z1)
    | ~ equidistant(X,V,X1,V1)
    | ~ equidistant(Y,V,Y1,V1)
    | ~ between(X,Y,Z)
    | ~ between(X1,Y1,Z1)
    | X = Y
    | equidistant(Z,V,Z1,V1) )).

%----A6 - Identity axiom for betweenness
cnf(identity_for_betweeness,axiom,
    ( ~ between(X,Y,X)
    | X = Y )).

%----A7 - Inner Pasch axiom, two clauses.
%----A7.1
cnf(inner_pasch1,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
    | between(V,inner_pasch(U,V,W,X,Y),Y) )).

%----A7.2
cnf(inner_pasch2,axiom,
    ( ~ between(U,V,W)
    | ~ between(Y,X,W)
```

```
    | between(X,inner_pasch(U,V,W,X,Y),U) )).

%----A8 - Lower dimension axiom, three clauses.
%----A8.1
cnf(lower_dimension1,axiom,
    ( ~
between(lower_dimension_point_1,lower_dimension_point_2,lower_dimension
_point_3) )).

%----A8.2
cnf(lower_dimension2,axiom,
    ( ~
between(lower_dimension_point_2,lower_dimension_point_3,lower_dimension
_point_1) )).

%----A8.3
cnf(lower_dimension3,axiom,
    ( ~
between(lower_dimension_point_3,lower_dimension_point_1,lower_dimension
_point_2) )).

%----A9 - Upper dimension axiom
cnf(upper_dimension,axiom,
    ( ~ equidistant(X,W,X,V)
    | ~ equidistant(Y,W,Y,V)
    | ~ equidistant(Z,W,Z,V)
    | between(X,Y,Z)
    | between(Y,Z,X)
    | between(Z,X,Y)
    | W = V )).

%----A10 - Euclid's axiom, three clauses.
%----A10.1
cnf(euclid1,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,V,euclid1(U,V,W,X,Y)) )).

%----A10.2
cnf(euclid2,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(U,X,euclid2(U,V,W,X,Y)) )).

%----A10.3
cnf(euclid3,axiom,
    ( ~ between(U,W,Y)
    | ~ between(V,W,X)
    | U = W
    | between(euclid1(U,V,W,X,Y),Y,euclid2(U,V,W,X,Y)) )).

%----A11 - Negate Weakened continuity axiom, two clauses.
%----A11.1
cnf(continuity1,axiom,
    ( ~ equidistant(U,V,U,V1)
```

```
    | ~ equidistant(U,X,U,X1)
    | ~ between(U,V,X)
    | ~ between(V,W,X)
    | between(V1,continuous(U,V,V1,W,X,X1),X1) )).

%---- negate A11.2
cnf(neg_continuity2,axiom,
    (  equidistant(U,V,U,V1)
    |  equidistant(U,X,U,X1)
    |  between(U,V,X)
    |  between(V,W,X)
    | ~  equidistant(U,W,U,continuous(U,V,V1,W,X,X1)) )).

%--------------------------------------------------------------------------
----
```

Appendix 17, continued

Model produced by VAMPIRE for Negate A11.2.

```
TRYING [1]
Finite Model Found!
% SZS status Satisfiable for GEO002-0.cond_seg.negA11.2
% SZS output start FiniteModel for GEO002-0.cond_seg.negA11.2
tff(declare_$i,type,$i:$tType).
tff(declare_$i1,type,lower_dimension_point_1:$i).
tff(finite_domain,axiom,
      ! [X:$i] : (
         X = lower_dimension_point_1
      ) ).

tff(declare_lower_dimension_point_2,type,lower_dimension_point_2:$i).
tff(lower_dimension_point_2_definition,axiom,lower_dimension_point_2 =
lower_dimension_point_1).
tff(declare_lower_dimension_point_3,type,lower_dimension_point_3:$i).
tff(lower_dimension_point_3_definition,axiom,lower_dimension_point_3 =
lower_dimension_point_1).
tff(declare_extension,type,extension: $i * $i * $i * $i > $i).
tff(function_extension,axiom,

extension(lower_dimension_point_1,lower_dimension_point_1,lower_dimensi
on_point_1,lower_dimension_point_1) = lower_dimension_point_1

).

tff(declare_inner_pasch,type,inner_pasch: $i * $i * $i * $i * $i > $i).
tff(function_inner_pasch,axiom,

inner_pasch(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid1,type,euclid1: $i * $i * $i * $i * $i > $i).
tff(function_euclid1,axiom,

euclid1(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1

).

tff(declare_euclid2,type,euclid2: $i * $i * $i * $i * $i > $i).
tff(function_euclid2,axiom,

euclid2(lower_dimension_point_1,lower_dimension_point_1,lower_dimension
_point_1,lower_dimension_point_1,lower_dimension_point_1) =
lower_dimension_point_1
```

```
).

tff(declare_continuous,type,continuous: $i * $i * $i * $i * $i * $i >
$i).
tff(function_continuous,axiom,

continuous(lower_dimension_point_1,lower_dimension_point_1,lower_dimens
ion_point_1,lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1) = lower_dimension_point_1

).

tff(declare_equidistant,type,equidistant: $i * $i * $i * $i > $o ).
tff(predicate_equidistant,axiom,

equidistant(lower_dimension_point_1,lower_dimension_point_1,lower_dimen
sion_point_1,lower_dimension_point_1)

).

tff(declare_between,type,between: $i * $i * $i > $o ).
tff(predicate_between,axiom,

~between(lower_dimension_point_1,lower_dimension_point_1,lower_dimensio
n_point_1)

).

% SZS output end FiniteModel for GEO002-0.cond_seg.negA11.2
% ------------------------------
% Version: Vampire 4.5.1 (commit 57a6f78c on 2020-07-15 11:59:04 +0200)
% Termination reason: Satisfiable

% Memory used [KB]: 4989
% Time elapsed: 0.002 s
% ------------------------------
% ------------------------------
```

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