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AN AUTOMATED EQUATIONAL LOGIC DERIVATION OF THE IMPLICATIONAL EQUIVALENCE OF THE PRINCIPIA MATHEMATICA AND ŁUKASIEWICZ'S *CN* SENTENTIAL CALCULI

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Abstract
Two logics are implicationally equivalent if the axioms and inference rules of each imply the axioms of the other. Using the automated equational logic deduction system contained in Mathematica , I show the implicational equivalence of the Principia Mathematica (PM) and Łukasiewicz's CN sentential calculi. The proof appears to be novel.

1.0 Introduction

Two logics are implicationally equivalent if the axioms and inference rules of each imply the axioms of the other.

Section 2 contains a summary of an automated equational logic derivation of the implicational equivalence of Łukasiewicz's CN sentential calculus ([1]; hereafter abbreviated "CN") and the sentential calculus of *Principia Mathematica* ([2]; hereafter abbreviated "PM")

1.1 Some terminology

I assume the definitions of term, value of a term, variable, and constant contained in [4], Chapter 3.

A *rewriting system* is a system of R rules that transforms expressions that satisfy some well-defined set of formation rules to another expression that satisfy those formation rules. For the purposes of this paper, I restrict "rewriting system" to a rewriting system that concerns identities of terms.

Two terms are said to be *identical* if the values of the terms are equal for all values of variables occurring in them.

A *reduction of a term T to a term T'* is a (typically recursive) rewriting of T to T' using a set of rewriting rules R such that T' is "simpler than" T (given some definition of "simpler than"). A *reduction sequence of a term T to a term T'* is a sequence T0 = T, T1, T2, T3, ..., Tn = T', where each Ti is the result of applying R to Ti-1, i = 1, 2, ..., n.

If "simpler than" is a partial ordering ([5], Df. 21, 72) on a reduction sequence that begins with T and ends with T' in a system with a set of rewriting rules R, "simpler than" induces a reduction order ([4], 102) on the reduction sequence that begins with T and ends with T'. A term *Tn is in normal form* if no application of R to Tn changes Tn.

A rewriting system is said to be *finitely terminating* if every reduction sequence of any term T produces, in a finite number of iterations, a normal form of T. A rewriting system is said to be *confluent* if the normal forms of all terms in the system are unique.

Some term rewriting systems are both finitely terminating and confluent ([4], esp. Chapter 9). Such rewriting systems have unique normal forms for all expressions. This permits us to use the the output of such a system to determine whether there is an identity between two terms T1 and T2 in the following manner. If T1 and T2 and have the same normal form, then there is an identity between T1 and T2. Otherwise, there is not an identity.

1.2 Mathematica's equational logic inference algorithm

The inference algorithm in Mathematica's ADF is the Knuth-Bendix completion algorithm ([6]). KBC attempts to transform a given finite set of identities (an "input" to KBC) to a finitely terminating, confluent term rewriting system that preserves identity. At initialization, KBC attempts to "orient" the identities supplied in its input according to the KnuthBendix reduction order ([4], Section 5.4.4). This results in an initial set of reduction rules. KBC then attempts to complete this initial set of rules with additional rules, obtaining their normal forms, and adding a new rule for every pair of the normal forms in accordance with the reduction order.

KBC may

1. Terminate with success, yielding a finitely terminating, confluent set of rules, or
2. Terminate with failure, or
3. Loop without terminating.

2.0 Proof of "The CN and PM sentential calculi are implicationally equivalent"

The *Mathematica* ([3]) script shown in this section was executed on a Dell Inspiron 545 with an Intel Core2 Quad CPU Q8200 @ 2.33 GHz and 8.00 GB RAM, running under Windows 10.

cn1 - cn3 are the axioms of Łukasiewicz's sentential calculus, expressed in *Mathematica* notation.

```
In[1]:= cn1 = ForAll[{x, y, z}, Implies[Implies[x, y], Implies[Implies[y, z], Implies[x, z]]]]
Out[1]= ∀{x,y,z} ((x ⇒ y) ⇒ ((y ⇒ z) ⇒ (x ⇒ z)))

In[2]:= cn2 = ForAll[{x, y}, Implies[x, Implies[Not[x], y]] ]
Out[2]= ∀{x,y} (x ⇒ (! x ⇒ y))

In[3]:= cn3 = ForAll[x, Implies[Implies[Not[x], x], x] ]
Out[3]= ∀x ((! x ⇒ x) ⇒ x)

In[4]:= CNAxioms = {cn1, cn2, cn3}
Out[4]= {∀{x,y,z} ((x ⇒ y) ⇒ ((y ⇒ z) ⇒ (x ⇒ z))), ∀{x,y} (x ⇒ (! x ⇒ y)), ∀x ((! x ⇒ x) ⇒ x)}
```

pm1 - pm5 are the axioms of the sentential calculus of Principia Mathematica, expressed in *Mathematica* notation.

```
In[5]:= pm1 = ForAll[{x, y}, Implies[y, Implies[Not[x], y]]]
Out[5]= ∀{x,y} (y ⇒ (! x ⇒ y))

In[6]:= pm2 = ForAll[x, Implies[Implies[Not[x], x], x]]
Out[6]= ∀x ((! x ⇒ x) ⇒ x)

In[7]:= pm3 = ForAll[{x, y}, Implies[Implies[Not[x], y], Implies[Not[y], x]]]
Out[7]= ∀{x,y} ((! x ⇒ y) ⇒ (! y ⇒ x))

In[8]:= pm4 =
         ForAll[{x, y.z}, Implies[Implies[y, z], Implies[Implies[Not[x], y], Implies[Not[x], z]]]]
Out[8]= ∀{x} ((y ⇒ z) ⇒ ((! x ⇒ y) ⇒ (! x ⇒ z)))

In[9]:= pm5 = ForAll[{x, y, z},
          Implies[Implies[Not[x], Implies[Not[y], z]], Implies[Not[y], Implies[Not[x], z]]]]
Out[9]= ∀{x,y,z} ((! x ⇒ (! y ⇒ z)) ⇒ (! y ⇒ (! x ⇒ z)))

In[10]:= pmAxioms = {pm1, pm2, pm3, pm4, pm5}
Out[10]= {∀{x,y} (y ⇒ (! x ⇒ y)), ∀x ((! x ⇒ x) ⇒ x), ∀{x,y} ((! x ⇒ y) ⇒ (! y ⇒ x)),
          ∀{x} ((y ⇒ z) ⇒ ((! x ⇒ y) ⇒ (! x ⇒ z))), ∀{x,y,z} ((! x ⇒ (! y ⇒ z)) ⇒ (! y ⇒ (! x ⇒ z)))}
```

Proof that CN implies PM.

In[11]:=
```
proofCNimppm1 = FindEquationalProof[pm1, CNAxioms]
```

Out[11]= ProofObject[Logic: Predicate/EquationalLogic Steps: 51
Theorem: $\forall_{\{x,y\}}$ (y ⇒ (!x ⇒ y))]

In[12]:=
```
proofCNimppm2 = FindEquationalProof[pm2, CNAxioms]
```

Out[12]= ProofObject[Logic: Predicate/EquationalLogic Steps: 17
Theorem: $\forall_{x}$ ((!x ⇒ x) ⇒ x)]

In[13]:=
```
proofCNimppm3 = FindEquationalProof[pm3, CNAxioms]
```

Out[13]= ProofObject[Logic: Predicate/EquationalLogic Steps: 49
Theorem: $\forall_{\{x,y\}}$ ((!x ⇒ y) ⇒ (!y ⇒ x))]

In[14]:=
```
proofCNimppm4 = FindEquationalProof[pm4, CNAxioms]
```

Out[14]= ProofObject[Logic: Predicate/EquationalLogic Steps: 69
Theorem: $\forall_{\{x\}}$ ((y ⇒ z) ⇒ ((!x ⇒ y) ⇒ (!x ⇒ z)))]

In[15]:=
```
proofCNimppm5 = FindEquationalProof[pm5, CNAxioms]
```

Out[15]= ProofObject[Logic: Predicate/EquationalLogic Steps: 87
Theorem: $\forall_{\{x,y,z\}}$ ((!x ⇒ (!y ⇒ z)) ⇒ (!y ⇒ (!x ⇒ z)))]

Proof of PM implies CN.

In[16]:=
```
proofPMimpCN1 = FindEquationalProof[cn1, PMAxioms]
```

Out[16]= ProofObject[Logic: Predicate/EquationalLogic Steps: 98
Theorem: $\forall_{\{x,y,z\}}$ ((x ⇒ y) ⇒ ((y ⇒ z) ⇒ (x ⇒ z)))]

In[17]:=
```
proofPMimpCN2 = FindEquationalProof[cn2, PMAxioms]
```

Out[17]= ProofObject[Logic: Predicate/EquationalLogic Steps: 36
Theorem: $\forall_{\{x,y\}}$ (x ⇒ (!x ⇒ y))]

In[18]:=
```
proofPMimpCN3 = FindEquationalProof[cn3, PMAxioms]
```

Out[18]= ProofObject[Logic: Predicate/EquationalLogic Steps: 24
Theorem: $\forall_{x}$ ((!x ⇒ x) ⇒ x)]

APPENDIX 1. Proof of CN implies PM1.

In[19]:=
```
proofCNimppm1["ProofNotebook"]
```

Axiom 1

We are given that:

$\forall_{\{x,y,z\}}((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)))$

Axiom 2

We are given that:

$\forall_{\{x,y\}}(x \Rightarrow (!x \Rightarrow y))$

Axiom 3

We are given that:

$\forall_{x}((!x \Rightarrow x) \Rightarrow x)$

Hypothesis 1

We would like to show that:

$\forall_{\{x,y\}}(y \Rightarrow (!x \Rightarrow y))$

Equationalized Axiom 1

We generate the "equationalized" axiom:

`x1 == (x1 || (x2 && !x2))`

Equationalized Axiom 2

We generate the "equationalized" axiom:

`x1 == (x1 && (x2 || !x2))`

Equationalized Axiom 3

We generate the "equationalized" axiom:

`(x1 || x2) == (x2 || x1)`

Equationalized Axiom 4

We generate the "equationalized" axiom:

`(x1 || (x2 && x3)) == ((x1 || x2) && (x1 || x3))`

Equationalized Axiom 5

We generate the "equationalized" axiom:

$(!x1 || x1 || x2) == (a_0 || !a_0)$

Equationalized Axiom 6

We generate the "equationalized" axiom:

$(!(!x1 || x2) || !(!x2 || x3) || !x1 || x3) == (a_0 || !a_0)$

Equationalized Axiom 7

We generate the "equationalized" axiom:

$(!(x1||x1)||x1) == (a_0||!a_0)$

Equationalized Axiom 8

We generate the "equationalized" axiom:

`((x1&&x2)||(x1&&x3)) == (x1&&(x2||x3))`

Equationalized Axiom 9

We generate the "equationalized" axiom:

`(x1&&x2) == (x2&&x1)`

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!y_0||x_1||y_0) == (a_0||!a_0)$

Critical Pair Lemma 1

The following expressions are equivalent:

`((x1&&!x1)||x2) == x2`

Proof

Note that the input for the rule:

`x1_||x2_↔x2_||x1_`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 3 and Equationalized Axiom 1 respectively.

Substitution Lemma 1

It can be shown that:

$(!(!x1||x2)||!(!x2||x3)||!x1||x3) == (!x_1||x_1||x_1)$

Proof

We start by taking Equationalized Axiom 6, and apply the substitution:

$a_0||!a_0 \to !x_1||x_1||x_1$

which follows from Equationalized Axiom 5.

Substitution Lemma 2

It can be shown that:

$(!(!x1||x2)||!(!x2||x3)||!x1||x3) == (!x_1||x_1||x_1)$

Proof

We start by taking Substitution Lemma 1, and apply the substitution:

`x1_||x2_→x2||x1`

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 3

It can be shown that:

`(x1||!(x1||x1)) == (a_0||!a_0)`

Proof

We start by taking Equationalized Axiom 7, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 4

It can be shown that:

`(x1||!(x1||x1)) == (a_0||!a_0)`

Proof

We start by taking Substitution Lemma 3, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 5

It can be shown that:

`(x1||!(x1||x1)) == (!x_1||x_1||x_1)`

Proof

We start by taking Substitution Lemma 4, and apply the substitution:

`a_0||!a_0→!x_1||x_1||x_1`

which follows from Equationalized Axiom 5.

Substitution Lemma 6

It can be shown that:

`(x1||!(x1||x1)) == (!x_1||x_1||x_1)`

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 7

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x_1||!(x_1||x_1)`

Proof

We start by taking Substitution Lemma 2, and apply the substitution:

`!x_1||x_1||x_1→x_1||!(x_1||x_1)`

which follows from Substitution Lemma 6.

Critical Pair Lemma 2

The following expressions are equivalent:

`(x1&&(x2||!x1))==(x1&&x2)`

Proof

Note that the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

contains a subpattern of the form:

`(x1_&&x2_)||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 8 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`((x1||!x1)&&x2)==x2`

Proof

Note that the input for the rule:

`x1_&&x2_↔x2_&&x1_`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 9 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 4

The following expressions are equivalent:

`(x1&&x2)==(x1&&(!x1||x2))`

Proof

Note that the input for the rule:

`(x1_&&!x1_)||x2_→x2`

contains a subpattern of the form:

`(x1_&&!x1_)||x2_`

which can be unified with the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 8 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

`(x1||x2)==(x1||(!x1&&x2))`

`(x1||x2)==(x1||(!x1&&x2))`

Proof

Note that the input for the rule:

`(x1_||!x1_)&&x2_→x2`

contains a subpattern of the form:

`(x1_||!x1_)&&x2_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

where these rules follow from Critical Pair Lemma 3 and Equationalized Axiom 4 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

`(x1&&x1)==x1`

Proof

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`x1_&&(!x1_||x2_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

`(x1&&(x1||x2))==(x1&&(a0||!a0))`

Proof

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`!x1_||x2_`

which can be unified with the input for the rule:

`"0"`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 5 respectively.

Substitution Lemma 8

It can be shown that:

`(x1&&(x1||x2))==x1`

Proof

We start by taking Critical Pair Lemma 7, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2

which follows from Equationalized Axiom 2.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1||x1)==x1`

Out[19]=

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`True`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x1_→x1`

where these rules follow from Critical Pair Lemma 5 and Critical Pair Lemma 6 respectively.

Substitution Lemma 9

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x1||!x1`

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

`x1_||x1_→x1`

which follows from Critical Pair Lemma 8.

Critical Pair Lemma 10

The following expressions are equivalent:

`x1==(x1&&(x2||x1))`

Proof

Note that the input for the rule:

`x1_&&(x1_||x2_)→x1`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_⟷x2_||x1_`

where these rules follow from Substitution Lemma 8 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 11

The following expressions are equivalent:

`(x1&&x1)==(x1&&(x1||!x1))`

Proof

Note that the input for the rule:

`x1_&&(x2_||!x1_)→x1&&x2`

contains a subpattern of the form:

`x2_||!x1_`

which can be unified with the input for the rule:

`x1_||!x1_→x1||!x1`

where these rules follow from Critical Pair Lemma 2 and Critical Pair Lemma 9 respectively.

Substitution Lemma 10

It can be shown that:

`(x1&&x1)==x1`

Proof

We start by taking Critical Pair Lemma 11, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2.

Substitution Lemma 11

It can be shown that:

`(x1&&x1)==x1`

Proof

We start by taking Substitution Lemma 10, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 9.

Substitution Lemma 12

It can be shown that:

`True`

Proof

We start by taking Substitution Lemma 11, and apply the substitution:

`x1_&&x1_→x1`

which follows from Critical Pair Lemma 6.

Critical Pair Lemma 12

The following expressions are equivalent:

`(!x1||x2)==(!x1||(x1&&x2))`

PROOF

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_`

which can be unified with the input for the rule:

`x1_→x1`

where these rules follow from Critical Pair Lemma 5 and Substitution Lemma 12 respectively.

Substitution Lemma 13

It can be shown that:

`(x1&&(x2||x1))==x1`

PROOF

We start by taking Critical Pair Lemma 10, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Substitution Lemma 14

It can be shown that:

`(x1&&(x1||x2))==x1`

PROOF

We start by taking Substitution Lemma 8, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Critical Pair Lemma 13

The following expressions are equivalent:

`(x1||!x1)==(x2||x1||!x1)`

PROOF

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x1_&&(x2_||x1_)`

which can be unified with the input for the rule:

`(x1_||!x1_)&&x2_→x2`

where these rules follow from Substitution Lemma 13 and Critical Pair Lemma 3 respectively.

Critical Pair Lemma 14

The following expressions are equivalent:

`(!x1||x1||x2) == (!x1||x1)`

PROOF

Note that the input for the rule:

`!x1_||(x1_&&x2_)→!x1||x2`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 12 and Substitution Lemma 14 respectively.

Critical Pair Lemma 15

The following expressions are equivalent:

`(x1||!x1) == (!(!x1||x2)||!x1||x1)`

PROOF

Note that the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[222,x1_||x1_→x1`

contains a subpattern of the form:

`!(!x2_||x3_)||!x1_||x3_`

which can be unified with the input for the rule:

`x1_||x2_||!x2_→x2||!x2`

where these rules follow from Substitution Lemma 9 and Critical Pair Lemma 13 respectively.

Substitution Lemma 15

It can be shown that:

`(x1||!x1) == (!x1||x1)`

PROOF

We start by taking Critical Pair Lemma 15, and apply the substitution:

`x1_||x2_||!x2_→x2||!x2`

which follows from Critical Pair Lemma 13.

Substitution Lemma 16

It can be shown that:

`(x1||!x1) == (!x1||x1)`

PROOF

We start by taking Substitution Lemma 15, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Substitution Lemma 17

It can be shown that:

$(x_1 || !x_1) == (x1 || !x1)$

PROOF

We start by taking Substitution Lemma 16, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 18

It can be shown that:

`(!x1||x1||x2) == (x1||!x1)`

PROOF

We start by taking Critical Pair Lemma 14, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 19

It can be shown that:

$(!y_0 || y_0 || x_1) == (a_0 || !a_0)$

PROOF

We start by taking Equationalized Hypothesis 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 20

It can be shown that:

$(!y_0 || y_0 || x_1) == (x_1 || !x_1)$

PROOF

We start by taking Substitution Lemma 19, and apply the substitution:

$x1\_ || !x1\_ \to x_1 || !x_1$

which follows from Substitution Lemma 17.

Substitution Lemma 21

It can be shown that:

$(y_0 || !y_0) == (x_1 || !x_1)$

PROOF

We start by taking Substitution Lemma 20, and apply the substitution:

`!x1_||x1_||x2_→x1||!x1`

which follows from Substitution Lemma 18.

Conclusion 1

We obtain the conclusion:

```
True
```

Proof

Take Substitution Lemma 21, and apply the substitution:

```
x1_||!x1_→x1||!x1
```

which follows from Substitution Lemma 17.

APPENDIX 2. Proof of CN implies PM2.

In[20]:= proofCNimppm2["ProofNotebook"]

Axiom 1

We are given that:

$\forall_{\{x,y,z\}}((x\Rightarrow y)\Rightarrow((y\Rightarrow z)\Rightarrow(x\Rightarrow z)))$

Axiom 2

We are given that:

$\forall_{\{x,y\}}(x\Rightarrow(!x\Rightarrow y))$

Axiom 3

We are given that:

$\forall_{x}((!x\Rightarrow x)\Rightarrow x)$

Hypothesis 1

We would like to show that:

$\forall_{x}((!x\Rightarrow x)\Rightarrow x)$

Equationalized Axiom 1

We generate the "equationalized" axiom:

(x1 || x2) == (x2 || x1)

Equationalized Axiom 2

We generate the "equationalized" axiom:

(!x1 || x1 || x2) == (a₀ || !a₀)

Equationalized Axiom 3

We generate the "equationalized" axiom:

(!(x1 || x1) || x1) == (a₀ || !a₀)

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!(x_0 || x_0) || x_0) == (a_0 || !a_0)$

Substitution Lemma 1

It can be shown that:

$(x1 || !(x1 || x1)) == (a_0 || !a_0)$

PROOF

We start by taking Equationalized Axiom 3, and apply the substitution:

x1_ || x2_ → x2 || x1

which follows from Equationalized Axiom 1.

Substitution Lemma 2

It can be shown that:

$(x1||!(x1||x1)) == (a_0||!a_0)$

Proof

We start by taking Substitution Lemma 1, and apply the substitution:

$x1\_||x2\_ \to x2||x1$

which follows from Equationalized Axiom 1.

Substitution Lemma 3

It can be shown that:

$(x1||!(x1||x1)) == (!x_0||x_0||x_0)$

Proof

Out[20]=

We start by taking Substitution Lemma 2, and apply the substitution:

$a_0||!a_0 \to !x_0||x_0||x_0$

which follows from Equationalized Axiom 2.

Substitution Lemma 4

It can be shown that:

$(x1||!(x1||x1)) == (!x_0||x_0||x_0)$

Proof

We start by taking Substitution Lemma 3, and apply the substitution:

$x1\_||x2\_ \to x2||x1$

which follows from Equationalized Axiom 1.

Substitution Lemma 5

It can be shown that:

$(!(x_0||x_0)||x_0) == (a_0||!a_0)$

Proof

We start by taking Equationalized Hypothesis 1, and apply the substitution:

$x1\_||x2\_ \to x2||x1$

which follows from Equationalized Axiom 1.

Substitution Lemma 6

It can be shown that:

$(x_0||!(x_0||x_0)) == (a_0||!a_0)$

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

$x1\_||x2\_ \to x2||x1$

which follows from Equationalized Axiom 1.

Substitution Lemma 7

It can be shown that:

```
(x0||!(x0||x0))==(!x0||x0||x0)
```

Proof

We start by taking Substitution Lemma 6, and apply the substitution:

```
a0||!a0→!x0||x0||x0
```

which follows from Equationalized Axiom 2.

Substitution Lemma 8

It can be shown that:

```
(x0||!(x0||x0))==(!x0||x0||x0)
```

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 1.

Conclusion 1

We obtain the conclusion:

```
True
```

Proof

Take Substitution Lemma 8, and apply the substitution:

```
x1_||!(x1_||x1_)→!x0||x0||x0
```

which follows from Substitution Lemma 4.

APPENDIX 3. Proof of CN implies PM3.

In[21]:= proofCNimppm3["ProofNotebook"]

Axiom 1

We are given that:

$\forall_{\{x,y,z\}}((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)))$

Axiom 2

We are given that:

$\forall_{\{x,y\}}(x \Rightarrow (!x \Rightarrow y))$

Axiom 3

We are given that:

$\forall_{x}((!x \Rightarrow x) \Rightarrow x)$

Hypothesis 1

We would like to show that:

$\forall_{\{x,y\}}((!x \Rightarrow y) \Rightarrow (!y \Rightarrow x))$

Equationalized Axiom 1

We generate the "equationalized" axiom:

x1 == (x1 || (x2 && !x2))

Equationalized Axiom 2

We generate the "equationalized" axiom:

x1 == (x1 && (x2 || !x2))

Equationalized Axiom 3

We generate the "equationalized" axiom:

(x1 || x2) == (x2 || x1)

Equationalized Axiom 4

We generate the "equationalized" axiom:

(x1 || (x2 && x3)) == ((x1 || x2) && (x1 || x3))

Equationalized Axiom 5

We generate the "equationalized" axiom:

(!x1 || x1 || x2) == (a_0 || !a_0)

Equationalized Axiom 6

We generate the "equationalized" axiom:

(!(!x1 || x2) || !(!x2 || x3) || !x1 || x3) == (a_0 || !a_0)

Equationalized Axiom 7

We generate the "equationalized" axiom:

$(!(x1||x1)||x1) == (a_0||!a_0)$

Equationalized Axiom 8

We generate the "equationalized" axiom:

((x1&&x2)||(x1&&x3)) == (x1&&(x2||x3))

Equationalized Axiom 9

We generate the "equationalized" axiom:

(x1&&x2) == (x2&&x1)

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!(x_1||y_0)||y_0||x_1) == (a_0||!a_0)$

Critical Pair Lemma 1

The following expressions are equivalent:

((x1&&!x1)||x2) == x2

Proof

Note that the input for the rule:

x1_||x2_ ↔ x2_||x1_

contains a subpattern of the form:

x1_||x2_

which can be unified with the input for the rule:

x1_||(x2_&&!x2_) → x1

where these rules follow from Equationalized Axiom 3 and Equationalized Axiom 1 respectively.

Substitution Lemma 1

It can be shown that:

$(!(!x1||x2)||!(!x2||x3)||!x1||x3) == (!x_1||x_1||x_1)$

Proof

We start by taking Equationalized Axiom 6, and apply the substitution:

$a_0||!a_0 \to !x_1||x_1||x_1$

which follows from Equationalized Axiom 5.

Substitution Lemma 2

It can be shown that:

$(!(!x1||x2)||!(!x2||x3)||!x1||x3) == (!x_1||x_1||x_1)$

Proof

We start by taking Substitution Lemma 1, and apply the substitution:

x1_||x2_ → x2||x1

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 3

It can be shown that:

$(x1||!(x1||x1)) == (a_0||!a_0)$

Proof

We start by taking Equationalized Axiom 7, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 4

It can be shown that:

$(x1||!(x1||x1)) == (a_0||!a_0)$

Proof

We start by taking Substitution Lemma 3, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 5

It can be shown that:

$(x1||!(x1||x1)) == (!x_1||x_1||x_1)$

Proof

We start by taking Substitution Lemma 4, and apply the substitution:

$a_0||!a_0 \to !x_1||x_1||x_1$

which follows from Equationalized Axiom 5.

Substitution Lemma 6

It can be shown that:

$(x1||!(x1||x1)) == (!x_1||x_1||x_1)$

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 7

It can be shown that:

$!(!x1\_||x2\_)||!(!x2\_||x3\_)||!x1\_||x3\_ \to x_1||!(x_1||x_1)$

Proof

We start by taking Substitution Lemma 2, and apply the substitution:

$!x_1||x_1||x_1 \to x_1||!(x_1||x_1)$

[illegible]

which follows from Substitution Lemma 6.

Critical Pair Lemma 2

The following expressions are equivalent:

`(x1&&(x2||!x1)) == (x1&&x2)`

Proof

Note that the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

contains a subpattern of the form:

`(x1_&&x2_)||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 8 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`((x1||!x1)&&x2) == x2`

Proof

Note that the input for the rule:

`x1_&&x2_↔x2_&&x1_`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 9 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 4

The following expressions are equivalent:

`(x1&&x2) == (x1&&(!x1||x2))`

Proof

Note that the input for the rule:

`(x1_&&!x1_)||x2_→x2`

contains a subpattern of the form:

`(x1_&&!x1_)||x2_`

which can be unified with the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 8 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

`(x1||x2) == (x1||(!x1&&x2))`

`(x1||x2)==(x1||(!x1&&x2))`

PROOF

Note that the input for the rule:

`(x1_||!x1_)&&x2_→x2`

contains a subpattern of the form:

`(x1_||!x1_)&&x2_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

where these rules follow from Critical Pair Lemma 3 and Equationalized Axiom 4 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

`(x1&&x1)==x1`

PROOF

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`x1_&&(!x1_||x2_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

`(x1&&(x1||x2))==(x1&&(a0||!a0))`

PROOF

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`!x1_||x2_`

which can be unified with the input for the rule:

`"0"`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 5 respectively.

Out[21]=

Substitution Lemma 8

It can be shown that:

`(x1&&(x1||x2))==x1`

PROOF

We start by taking Critical Pair Lemma 7, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2

which follows from Equationalized Axiom 2.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1||x1)==x1`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`True`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x1_→x1`

where these rules follow from Critical Pair Lemma 5 and Critical Pair Lemma 6 respectively.

Substitution Lemma 9

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x1||!x1`

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

`x1_||x1_→x1`

which follows from Critical Pair Lemma 8.

Critical Pair Lemma 10

The following expressions are equivalent:

`x1==(x1&&(x2||x1))`

Proof

Note that the input for the rule:

`x1_&&(x1_||x2_)→x1`

contains a subpattern of the form:

```
x1_||x2_
```

which can be unified with the input for the rule:

```
x1_||x2_↔x2_||x1_
```

where these rules follow from Substitution Lemma 8 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 11

The following expressions are equivalent:

```
(x1&&x1)==(x1&&(x1||!x1))
```

Proof

Note that the input for the rule:

```
x1_&&(x2_||!x1_)→x1&&x2
```

contains a subpattern of the form:

```
x2_||!x1_
```

which can be unified with the input for the rule:

```
x1_||!x1_→x1||!x1
```

where these rules follow from Critical Pair Lemma 2 and Critical Pair Lemma 9 respectively.

Substitution Lemma 10

It can be shown that:

```
(x1&&x1)==x1
```

Proof

We start by taking Critical Pair Lemma 11, and apply the substitution:

```
x1_&&(x2_||!x2_)→x1
```

which follows from Equationalized Axiom 2.

Substitution Lemma 11

It can be shown that:

```
(x1&&x1)==x1
```

Proof

We start by taking Substitution Lemma 10, and apply the substitution:

```
x1_&&x2_→x2&&x1
```

which follows from Equationalized Axiom 9.

Substitution Lemma 12

It can be shown that:

```
True
```

Proof

We start by taking Substitution Lemma 11, and apply the substitution:

```
x1_&&x1_→x1
```

which follows from Critical Pair Lemma 6.

Substitution Lemma 13

It can be shown that:

`(x1&&(x2||x1))==x1`

PROOF

We start by taking Critical Pair Lemma 10, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Critical Pair Lemma 12

The following expressions are equivalent:

`(x1||!x1)==(x2||x1||!x1)`

PROOF

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x1_&&(x2_||x1_)`

which can be unified with the input for the rule:

`(x1_||!x1_)&&x2_→x2`

where these rules follow from Substitution Lemma 13 and Critical Pair Lemma 3 respectively.

Critical Pair Lemma 13

The following expressions are equivalent:

$(x_1||!x_1)$`==(!(!x1||x2)||!x1||x1)`

PROOF

Note that the input for the rule:

`` Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[222,x1_||x1_→x1 ``

contains a subpattern of the form:

`!(!x2_||x3_)||!x1_||x3_`

which can be unified with the input for the rule:

`x1_||x2_||!x2_→x2||!x2`

where these rules follow from Substitution Lemma 9 and Critical Pair Lemma 12 respectively.

Substitution Lemma 14

It can be shown that:

$(x_1||!x_1)$`==(!x1||x1)`

PROOF

We start by taking Critical Pair Lemma 13, and apply the substitution:

`x1_||x2_||!x2_→x2||!x2`

which follows from Critical Pair Lemma 12.

Substitution Lemma 15

It can be shown that:

$(x_1 || !x_1) == (!x1 || x1)$

PROOF

We start by taking Substitution Lemma 14, and apply the substitution:

x1_→x1

which follows from Substitution Lemma 12.

Substitution Lemma 16

It can be shown that:

$(x_1 || !x_1) == (x1 || !x1)$

PROOF

We start by taking Substitution Lemma 15, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 17

It can be shown that:

$(!(y_0 || x_1) || y_0 || x_1) == (a_0 || !a_0)$

PROOF

We start by taking Equationalized Hypothesis 1, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 18

It can be shown that:

$(y_0 || x_1 || !(y_0 || x_1)) == (a_0 || !a_0)$

PROOF

We start by taking Substitution Lemma 17, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 19

It can be shown that:

$(y_0 || x_1 || !(y_0 || x_1)) == (x_1 || !x_1)$

PROOF

We start by taking Substitution Lemma 18, and apply the substitution:

x1_||!x1_→$x_1 || !x_1$

which follows from Substitution Lemma 16.

Substitution Lemma 20

It can be shown that:

`(y0||x1||!(y0||x1))==(x1||!x1)`

PROOF

We start by taking Substitution Lemma 19, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Substitution Lemma 21

It can be shown that:

`(y0||x1||!(y0||x1))==(x1||!x1)`

PROOF

We start by taking Substitution Lemma 20, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 12.

Conclusion 1

We obtain the conclusion:

`True`

PROOF

Take Substitution Lemma 21, and apply the substitution:

`x1_||!x1_→x1||!x1`

which follows from Substitution Lemma 16.

APPENDIX 4. Proof of CN implies PM4.

In[22]:=
```
proofCNimppm4["ProofNotebook"]
```

Axiom 1

We are given that:

$\forall_{\{x,y,z\}}((x\Rightarrow y)\Rightarrow((y\Rightarrow z)\Rightarrow(x\Rightarrow z)))$

Axiom 2

We are given that:

$\forall_{\{x,y\}}(x\Rightarrow(!x\Rightarrow y))$

Axiom 3

We are given that:

$\forall_{x}((!x\Rightarrow x)\Rightarrow x)$

Hypothesis 1

We would like to show that:

$\forall_{\{x\}}((y\Rightarrow z)\Rightarrow((!x\Rightarrow y)\Rightarrow(!x\Rightarrow z)))$

Equationalized Axiom 1

We generate the "equationalized" axiom:

`x1==(x1||(x2&&!x2))`

Equationalized Axiom 2

We generate the "equationalized" axiom:

`x1==(x1&&(x2||!x2))`

Equationalized Axiom 3

We generate the "equationalized" axiom:

`(x1||x2)==(x2||x1)`

Equationalized Axiom 4

We generate the "equationalized" axiom:

`(x1||(x2&&x3))==((x1||x2)&&(x1||x3))`

Equationalized Axiom 5

We generate the "equationalized" axiom:

$(!x1||x1||x2)==(a_0||!a_0)$

Equationalized Axiom 6

We generate the "equationalized" axiom:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(a0||!a0)`

Equationalized Axiom 7

We generate the "equationalized" axiom:

`(!(x1||x1)||x1)==(a0||!a0)`

Equationalized Axiom 8

We generate the "equationalized" axiom:

`((x1&&x2)||(x1&&x3))==(x1&&(x2||x3))`

Equationalized Axiom 9

We generate the "equationalized" axiom:

`(x1&&x2)==(x2&&x1)`

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

`(!(!y||z)||!(x0||y)||x0||z)==(a0||!a0)`

Critical Pair Lemma 1

The following expressions are equivalent:

`((x1&&!x1)||x2)==x2`

Proof

Note that the input for the rule:

`x1_||x2_↔x2_||x1_`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 3 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 2

The following expressions are equivalent:

`(x1||(x2&&!x1))==(x1||x2)`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`(x1_||x2_)&&(x1_||x3_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 4 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`(x1||(x2&&x3))==((x2||x1)&&(x1||x3))`

PROOF

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Equationalized Axiom 4 and Equationalized Axiom 3 respectively.

Substitution Lemma 1

It can be shown that:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(!x0||x0||x0)`

PROOF

We start by taking Equationalized Axiom 6, and apply the substitution:

`a0||!a0→!x0||x0||x0`

which follows from Equationalized Axiom 5.

Substitution Lemma 2

It can be shown that:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(!x0||x0||x0)`

PROOF

We start by taking Substitution Lemma 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 3

It can be shown that:

`(x1||!(x1||x1))==(a0||!a0)`

PROOF

We start by taking Equationalized Axiom 7, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 4

It can be shown that:

`(x1||!(x1||x1))==(a0||!a0)`

PROOF

We start by taking Substitution Lemma 3, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Substitution Lemma 5

It can be shown that:

```
(x1||!(x1||x1))==(!x0||x0||x0)
```

Proof

We start by taking Substitution Lemma 4, and apply the substitution:

```
a0||!a0→!x0||x0||x0
```

which follows from Equationalized Axiom 5.

Substitution Lemma 6

It can be shown that:

```
(x1||!(x1||x1))==(!x0||x0||x0)
```

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Substitution Lemma 7

It can be shown that:

```
!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x0||!(x0||x0)
```

Proof

We start by taking Substitution Lemma 2, and apply the substitution:

```
!x0||x0||x0→x0||!(x0||x0)
```

which follows from Substitution Lemma 6.

Critical Pair Lemma 4

The following expressions are equivalent:

```
(x1&&(x2||!x1))==(x1&&x2)
```

Proof

Note that the input for the rule:

```
(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)
```

contains a subpattern of the form:

```
(x1_&&x2_)||(x1_&&x3_)
```

which can be unified with the input for the rule:

```
x1_||(x2_&&!x2_)→x1
```

where these rules follow from Equationalized Axiom 8 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

The following expressions are equivalent:

`((x1||!x1)&&x2)==x2`

PROOF

Note that the input for the rule:

`x1_&&x2_↔x2_&&x1_`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 9 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

`(x1&&x2)==(x1&&(!x1||x2))`

PROOF

Note that the input for the rule:

`(x1_&&!x1_)||x2_→x2`

contains a subpattern of the form:

`(x1_&&!x1_)||x2_`

which can be unified with the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 8 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

`(x1||x2)==(x1||(!x1&&x2))`

PROOF

Note that the input for the rule:

`(x1_||!x1_)&&x2_→x2`

contains a subpattern of the form:

`(x1_||!x1_)&&x2_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 4 respectively.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1||x1)==x1`

PROOF

Note that the input for the rule:

`x1_||(x2_&&!x1_)→x1||x2`

contains a subpattern of the form:

`x1_||(x2_&&!x1_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 2 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`(x1||(x2&&x1))==((x1||x2)&&x1)`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x3_`

which can be unified with the input for the rule:

`x1_||x1_→x1`

where these rules follow from Equationalized Axiom 4 and Critical Pair Lemma 8 respectively.

Substitution Lemma 8

It can be shown that:

`(x1||(x2&&x1))==(x1&&(x1||x2))`

Proof

We start by taking Critical Pair Lemma 9, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 9.

Critical Pair Lemma 10

The following expressions are equivalent:

`(x1&&x1)==x1`

Proof

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`x1_&&(!x1_||x2_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 11

The following expressions are equivalent:

`(x1&&(x1||x2))==(x1&&(a0||!a0))`

`(x1&&(x1||x2))==(x1&&("0"||!"0"))`

Proof

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`!x1_||x2_`

which can be unified with the input for the rule:

`"0"`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 5 respectively.

Substitution Lemma 9

It can be shown that:

`(x1&&(x1||x2))==x1`

Proof

We start by taking Critical Pair Lemma 11, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2.

Critical Pair Lemma 12

The following expressions are equivalent:

`(x1||x1)==x1`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 7 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 13

The following expressions are equivalent:

`True`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x1_→x1`

where these rules follow from Critical Pair Lemma 7 and Critical Pair Lemma 10 respectively.

where these rules follow from Critical Pair Lemma 7 and Critical Pair Lemma 10 respectively.

Substitution Lemma 10

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x0||!x0`

PROOF

We start by taking Substitution Lemma 7, and apply the substitution:

`x1_||x1_→x1`

which follows from Critical Pair Lemma 12.

Critical Pair Lemma 14

The following expressions are equivalent:

`x1==(x1&&(x2||x1))`

PROOF

Note that the input for the rule:

`x1_&&(x1_||x2_)→x1`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

Out[22]= `x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 9 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 15

The following expressions are equivalent:

`(x1&&x1)==(x1&&(x1||!x1))`

PROOF

Note that the input for the rule:

`x1_&&(x2_||!x1_)→x1&&x2`

contains a subpattern of the form:

`x2_||!x1_`

which can be unified with the input for the rule:

`x1_||!x1_→x1||!x1`

where these rules follow from Critical Pair Lemma 4 and Critical Pair Lemma 13 respectively.

Substitution Lemma 11

It can be shown that:

`(x1&&x1)==x1`

PROOF

We start by taking Critical Pair Lemma 15, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2.

Substitution Lemma 12

It can be shown that:

`(x1&&x1)==x1`

Proof

We start by taking Substitution Lemma 11, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 9.

Substitution Lemma 13

It can be shown that:

`True`

Proof

We start by taking Substitution Lemma 12, and apply the substitution:

`x1_&&x1_→x1`

which follows from Critical Pair Lemma 10.

Substitution Lemma 14

It can be shown that:

`(x1&&(x2||x1))==x1`

Proof

We start by taking Critical Pair Lemma 14, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Substitution Lemma 15

It can be shown that:

`(x1&&(x1||x2))==x1`

Proof

We start by taking Substitution Lemma 9, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Critical Pair Lemma 16

The following expressions are equivalent:

`(x1||!x1)==(x2||x1||!x1)`

Proof

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x1_&&(x2_||x1_)`

which can be unified with the input for the rule:

`(x1_||!x1_)&&x2_→x2`

where these rules follow from Substitution Lemma 14 and Critical Pair Lemma 5 respectively.

Substitution Lemma 16

It can be shown that:

`x1_||(x2_&&x1_)→x1`

Proof

We start by taking Substitution Lemma 8, and apply the substitution:

`x1_&&(x1_||x2_)→x1`

which follows from Substitution Lemma 15.

Critical Pair Lemma 17

The following expressions are equivalent:

`(x1||x2)==(x1||x2||x1)`

Proof

Note that the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[412,x1_&&(x1_||`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Substitution Lemma 16 and Substitution Lemma 15 respectively.

Substitution Lemma 17

It can be shown that:

`(x1||x2)==(x1||x1||x2)`

Proof

We start by taking Critical Pair Lemma 17, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Critical Pair Lemma 18

The following expressions are equivalent:

`(x1||((x1||x2)&&x3))==((x1||x2)&&(x1||x3))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x1_||x2_→x1||x2`

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 17 respectively.

Substitution Lemma 18

It can be shown that:

`(x1||((x1||x2)&&x3))==(x1||(x2&&x3))`

Proof

We start by taking Critical Pair Lemma 18, and apply the substitution:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

which follows from Equationalized Axiom 4.

Critical Pair Lemma 19

The following expressions are equivalent:

`(x0||!x0)==(!(!x1||x2)||!x1||x1)`

Proof

Note that the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[222,x1_||x1_→x1`

contains a subpattern of the form:

`!(!x2_||x3_)||!x1_||x3_`

which can be unified with the input for the rule:

`x1_||x2_||!x2_→x2||!x2`

where these rules follow from Substitution Lemma 10 and Critical Pair Lemma 16 respectively.

Substitution Lemma 19

It can be shown that:

`(x0||!x0)==(!x1||x1)`

Proof

We start by taking Critical Pair Lemma 19, and apply the substitution:

`x1_||x2_||!x2_→x2||!x2`

which follows from Critical Pair Lemma 16.

Substitution Lemma 20

It can be shown that:

`(x0||!x0)==(!x1||x1)`

Proof

We start by taking Substitution Lemma 19, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Substitution Lemma 21

It can be shown that:

It can be shown that:

`(x0||!x0)==(x1||!x1)`

PROOF

We start by taking Substitution Lemma 20, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Critical Pair Lemma 20

The following expressions are equivalent:

`(x1||(x2&&(x2||x3)))==(x1||x2||(x1&&x3))`

PROOF

Note that the input for the rule:

`x1_||((x1_||x2_)&&x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`(x1_||x2_)&&x3_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x2_||x3_)→x2||(x1&&x3)`

where these rules follow from Substitution Lemma 18 and Critical Pair Lemma 3 respectively.

Substitution Lemma 22

It can be shown that:

`(x1||x2)==(x1||x2||(x1&&x3))`

PROOF

We start by taking Critical Pair Lemma 20, and apply the substitution:

`x1_&&(x1_||x2_)→x1`

which follows from Substitution Lemma 15.

Critical Pair Lemma 21

The following expressions are equivalent:

`(x1||x2)==(x1||(x1&&x3)||x2)`

PROOF

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x2_||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 22 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 22

The following expressions are equivalent:

`(x1||x2) == (x1||x2||(x3&&x1))`

PROOF

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x1_&&x3_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Substitution Lemma 22 and Equationalized Axiom 9 respectively.

Critical Pair Lemma 23

The following expressions are equivalent:

`(x1||x2) == (x1||(x3&&x1)||x2)`

PROOF

Note that the input for the rule:

`x1_||(x1_&&x2_)||x3_→x1||x3`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 21 and Equationalized Axiom 9 respectively.

Critical Pair Lemma 24

The following expressions are equivalent:

`(x1||x2||x3) == (x1||x2||x3||x1)`

PROOF

Note that the input for the rule:

`x1_||x2_||(x3_&&x1_)→x1||x2`

contains a subpattern of the form:

`x3_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 22 and Substitution Lemma 15 respectively.

Critical Pair Lemma 25

The following expressions are equivalent:

`(x1||x2||x3) == (x1||x2||x2||x3)`

PROOF

Note that the input for the rule:

`x1_||(x2_&&x1_)||x3_→x1||x3`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x2_||x1_)→x1`

where these rules follow from Critical Pair Lemma 23 and Substitution Lemma 14 respectively.

Critical Pair Lemma 26

The following expressions are equivalent:

`(x1||x2||x3) == (x3||x1||x1||x2)`

Proof

Note that the input for the rule:

`x1_||x2_||x3_||x1_→x1||x2||x3`

contains a subpattern of the form:

`x1_||x2_||x3_||x1_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 24 and Equationalized Axiom 3 respectively.

Substitution Lemma 23

It can be shown that:

`(x1||x2||x3) == (x3||x1||x2)`

Proof

We start by taking Critical Pair Lemma 26, and apply the substitution:

`x1_||x2_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 25.

Substitution Lemma 24

It can be shown that:

$(!(!y||z)||x_0||z||!(x_0||y)) == (a_0||!a_0)$

Proof

We start by taking Equationalized Hypothesis 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 25

It can be shown that:

$(x_0||z||!(x_0||y)||!(!y||z)) == (a_0||!a_0)$

Proof

We start by taking Substitution Lemma 24, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 26

It can be shown that:

`(x0||z||!(x0||y)||!(!y||z))==(x0||!x0)`

Proof

We start by taking Substitution Lemma 25, and apply the substitution:

`x1_||!x1_→x0||!x0`

which follows from Substitution Lemma 21.

Substitution Lemma 27

It can be shown that:

`(!(!y||z)||x0||z||!(x0||y))==(x0||!x0)`

Proof

We start by taking Substitution Lemma 26, and apply the substitution:

`x1_||x2_||x3_→x3||x1||x2`

which follows from Substitution Lemma 23.

Substitution Lemma 28

It can be shown that:

`(!(x0||y)||!(!y||z)||x0||z)==(x0||!x0)`

Proof

We start by taking Substitution Lemma 27, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Conclusion 1

We obtain the conclusion:

`True`

Proof

Take Substitution Lemma 28, and apply the substitution:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→x0||!x0`

which follows from Substitution Lemma 10.

APPENDIX 5. Proof of CN implies PM5.

In[23]:= `proofCNimppm5["ProofNotebook"]`

Axiom 1

We are given that:

`∀{x,y,z} ((x⇒y)⇒((y⇒z)⇒(x⇒z)))`

Axiom 2

We are given that:

`∀{x,y} (x⇒(!x⇒y))`

Axiom 3

We are given that:

`∀x ((!x⇒x)⇒x)`

Hypothesis 1

We would like to show that:

`∀{x,y,z} ((!x⇒(!y⇒z))⇒(!y⇒(!x⇒z)))`

Equationalized Axiom 1

We generate the "equationalized" axiom:

`x1==(x1||(x2&&!x2))`

Equationalized Axiom 2

We generate the "equationalized" axiom:

`x1==(x1&&(x2||!x2))`

Equationalized Axiom 3

We generate the "equationalized" axiom:

`(x1||x2)==(x2||x1)`

Equationalized Axiom 4

We generate the "equationalized" axiom:

`(x1||(x2&&x3))==((x1||x2)&&(x1||x3))`

Equationalized Axiom 5

We generate the "equationalized" axiom:

`(!x1||x1||x2)==(a0||!a0)`

Equationalized Axiom 6

We generate the "equationalized" axiom:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(a0||!a0)`

Equationalized Axiom 7

We generate the "equationalized" axiom:

`(!(x1||x1)||x1)==(a0||!a0)`

Equationalized Axiom 8

We generate the "equationalized" axiom:

`((x1&&x2)||(x1&&x3))==(x1&&(x2||x3))`

Equationalized Axiom 9

We generate the "equationalized" axiom:

`(x1&&x2)==(x2&&x1)`

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

`(!(x2||y1||z0)||y1||x2||z0)==(a0||!a0)`

Critical Pair Lemma 1

The following expressions are equivalent:

`((x1&&!x1)||x2)==x2`

Proof

Note that the input for the rule:

`x1_||x2_↔x2_||x1_`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 3 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 2

The following expressions are equivalent:

`(x1||(x2&&!x1))==(x1||x2)`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`(x1_||x2_)&&(x1_||x3_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 4 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`(x1||(x2&&x3))==((x2||x1)&&(x1||x3))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Equationalized Axiom 4 and Equationalized Axiom 3 respectively.

Substitution Lemma 1

It can be shown that:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(!y1||y1||y1)`

Proof

We start by taking Equationalized Axiom 6, and apply the substitution:

`a0||!a0→!y1||y1||y1`

which follows from Equationalized Axiom 5.

Substitution Lemma 2

It can be shown that:

`(!(!x1||x2)||!(!x2||x3)||!x1||x3)==(!y1||y1||y1)`

Proof

We start by taking Substitution Lemma 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 3

It can be shown that:

`(x1||!(x1||x1))==(a0||!a0)`

Proof

We start by taking Equationalized Axiom 7, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 4

It can be shown that:

`(x1||!(x1||x1))==(a0||!a0)`

Proof

We start by taking Substitution Lemma 3, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 5

It can be shown that:

`(x1||!(x1||x1))==(!y1||y1||y1)`

Proof

We start by taking Substitution Lemma 4, and apply the substitution:

`a0||!a0→!y1||y1||y1`

which follows from Equationalized Axiom 5.

Substitution Lemma 6

It can be shown that:

`(x1||!(x1||x1))==(!y1||y1||y1)`

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 7

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→y1||!(y1||y1)`

Proof

We start by taking Substitution Lemma 2, and apply the substitution:

`!y1||y1||y1→y1||!(y1||y1)`

which follows from Substitution Lemma 6.

Critical Pair Lemma 4

The following expressions are equivalent:

`(x1&&(x2||!x1))==(x1&&x2)`

Proof

Note that the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

contains a subpattern of the form:

`(x1_&&x2_)||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Equationalized Axiom 8 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

`((x1||!x1)&&x2)==x2`

((x1||!x1)&&x2)→x2

PROOF

Note that the input for the rule:

`x1_&&x2_↔x2_&&x1_`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Equationalized Axiom 9 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

`(x1&&x2)==(x1&&(!x1||x2))`

PROOF

Note that the input for the rule:

`(x1_&&!x1_)||x2_→x2`

contains a subpattern of the form:

`(x1_&&!x1_)||x2_`

which can be unified with the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 8 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

`(x1||x2)==(x1||(!x1&&x2))`

PROOF

Note that the input for the rule:

`(x1_||!x1_)&&x2_→x2`

contains a subpattern of the form:

`(x1_||!x1_)&&x2_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 4 respectively.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1||!x1)==(x2||!x2)`

PROOF

Note that the input for the rule:

`(x1_||!x1_)&&x2_→x2`

contains a subpattern of the form:

contains a subpattern of the form:

`(x1_||!x1_)&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`(x1||x1)==x1`

Proof

Note that the input for the rule:

`x1_||(x2_&&!x1_)→x1||x2`

contains a subpattern of the form:

`x1_||(x2_&&!x1_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 2 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 10

The following expressions are equivalent:

`(x1||(x2&&x1))==((x1||x2)&&x1)`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x3_`

which can be unified with the input for the rule:

`x1_||x1_→x1`

where these rules follow from Equationalized Axiom 4 and Critical Pair Lemma 9 respectively.

Substitution Lemma 8

It can be shown that:

`(x1||(x2&&x1))==(x1&&(x1||x2))`

Proof

We start by taking Critical Pair Lemma 10, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 9.

Critical Pair Lemma 11

The following expressions are equivalent:

`(x1&&x1)==x1`

Proof

PROOF

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`x1_&&(!x1_||x2_)`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 12

The following expressions are equivalent:

`(x1&&(x1||x2))==(x1&&(a0||!a0))`

PROOF

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`!x1_||x2_`

which can be unified with the input for the rule:

`"0"`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 5 respectively.

Substitution Lemma 9

It can be shown that:

`(x1&&(x1||x2))==x1`

PROOF

We start by taking Critical Pair Lemma 12, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2.

Critical Pair Lemma 13

The following expressions are equivalent:

`(x1||x1)==x1`

PROOF

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 7 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 14

The following expressions are equivalent:

`True`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x1_→x1`

where these rules follow from Critical Pair Lemma 7 and Critical Pair Lemma 11 respectively.

Substitution Lemma 10

It can be shown that:

`!(!x1_||x2_)||!(!x2_||x3_)||!x1_||x3_→y1||!y1`

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

`x1_||x1_→x1`

which follows from Critical Pair Lemma 13.

Critical Pair Lemma 15

The following expressions are equivalent:

`x1==(x1&&(x2||x1))`

Proof

Note that the input for the rule:

`x1_&&(x1_||x2_)→x1`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 9 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 16

The following expressions are equivalent:

`(x1&&x1)==(x1&&(x1||!x1))`

Proof

Note that the input for the rule:

`x1_&&(x2_||!x1_)→x1&&x2`

contains a subpattern of the form:

`x2_||!x1_`

which can be unified with the input for the rule:

`x1_||!x1_→x1||!x1`

where these rules follow from Critical Pair Lemma 4 and Critical Pair Lemma 14 respectively.

Substitution Lemma 11

It can be shown that:

`(x1&&x1)==x1`

Proof

We start by taking Critical Pair Lemma 16, and apply the substitution:

`x1_&&(x2_||!x2_)→x1`

which follows from Equationalized Axiom 2.

Substitution Lemma 12

It can be shown that:

`(x1&&x1)==x1`

Proof

We start by taking Substitution Lemma 11, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 9.

Substitution Lemma 13

It can be shown that:

`True`

Proof

We start by taking Substitution Lemma 12, and apply the substitution:

`x1_&&x1_→x1`

which follows from Critical Pair Lemma 11.

Substitution Lemma 14

It can be shown that:

`(x1&&(x2||x1))==x1`

Proof

We start by taking Critical Pair Lemma 15, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Substitution Lemma 15

It can be shown that:

`(x1&&(x1||x2))==x1`

Proof

We start by taking Substitution Lemma 9, and apply the substitution:

```
x1_→x1
```

which follows from Substitution Lemma 13.

Critical Pair Lemma 17

The following expressions are equivalent:

```
(x1||!x1) == (x2||x1||!x1)
```

Proof

Note that the input for the rule:

```
x1_&&(x2_||x1_)→x1
```

contains a subpattern of the form:

Out[23]=

```
x1_&&(x2_||x1_)
```

which can be unified with the input for the rule:

```
(x1_||!x1_)&&x2_→x2
```

where these rules follow from Substitution Lemma 14 and Critical Pair Lemma 5 respectively.

Substitution Lemma 16

It can be shown that:

```
x1_||(x2_&&x1_)→x1
```

Proof

We start by taking Substitution Lemma 8, and apply the substitution:

```
x1_&&(x1_||x2_)→x1
```

which follows from Substitution Lemma 15.

Critical Pair Lemma 18

The following expressions are equivalent:

```
(x1||x2) == (x1||x2||x1)
```

Proof

Note that the input for the rule:

```
Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[412,x1_&&(x1_||
```

contains a subpattern of the form:

```
x2_&&x1_
```

which can be unified with the input for the rule:

```
x1_&&(x1_||x2_)→x1
```

where these rules follow from Substitution Lemma 16 and Substitution Lemma 15 respectively.

Substitution Lemma 17

It can be shown that:

```
(x1||x2) == (x1||x1||x2)
```

Proof

We start by taking Critical Pair Lemma 18, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Critical Pair Lemma 19

The following expressions are equivalent:

`(x1||((x1||x2)&&x3))==((x1||x2)&&(x1||x3))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x1_||x2_→x1||x2`

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 17 respectively.

Substitution Lemma 18

It can be shown that:

`(x1||((x1||x2)&&x3))==(x1||(x2&&x3))`

Proof

We start by taking Critical Pair Lemma 19, and apply the substitution:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

which follows from Equationalized Axiom 4.

Critical Pair Lemma 20

The following expressions are equivalent:

$(y_1 || !y_1)$ `==(!(!x1||x2)||!x1||x1)`

Proof

Note that the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[222,x1_||x1_→x1`

contains a subpattern of the form:

`!(!x2_||x3_)||!x1_||x3_`

which can be unified with the input for the rule:

`x1_||x2_||!x2_→x2||!x2`

where these rules follow from Substitution Lemma 10 and Critical Pair Lemma 17 respectively.

Substitution Lemma 19

It can be shown that:

$(y_1 || !y_1)$ `==(!x1||x1)`

Proof

We start by taking Critical Pair Lemma 20, and apply the substitution:

`x1_||x2_||!x2_→x2||!x2`

which follows from Critical Pair Lemma 17.

Substitution Lemma 20

It can be shown that:

`(y1 || !y1) == (!x1 || x1)`

PROOF

We start by taking Substitution Lemma 19, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Substitution Lemma 21

It can be shown that:

`(y1 || !y1) == (x1 || !x1)`

PROOF

We start by taking Substitution Lemma 20, and apply the substitution:

`x1_ || x2_→x2 || x1`

which follows from Equationalized Axiom 3.

Critical Pair Lemma 21

The following expressions are equivalent:

`(x1 || (x2 && (x2 || x3))) == (x1 || x2 || (x1 && x3))`

PROOF

Note that the input for the rule:

`x1_ || ((x1_ || x2_) && x3_)→x1 || (x2 && x3)`

contains a subpattern of the form:

`(x1_ || x2_) && x3_`

which can be unified with the input for the rule:

`(x1_ || x2_) && (x2_ || x3_)→x2 || (x1 && x3)`

where these rules follow from Substitution Lemma 18 and Critical Pair Lemma 3 respectively.

Substitution Lemma 22

It can be shown that:

`(x1 || x2) == (x1 || x2 || (x1 && x3))`

PROOF

We start by taking Critical Pair Lemma 21, and apply the substitution:

`x1_ && (x1_ || x2_)→x1`

which follows from Substitution Lemma 15.

Critical Pair Lemma 22

The following expressions are equivalent:

`(x1 || x2) == (x1 || (x1 && x3) || x2)`

Proof

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x2_||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 22 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 23

The following expressions are equivalent:

`(x1||x2) == (x1||x2||(x3&&x1))`

Proof

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x1_&&x3_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Substitution Lemma 22 and Equationalized Axiom 9 respectively.

Critical Pair Lemma 24

The following expressions are equivalent:

`(x1||x2) == (x1||(x3&&x1)||x2)`

Proof

Note that the input for the rule:

`x1_||(x1_&&x2_)||x3_→x1||x3`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 22 and Equationalized Axiom 9 respectively.

Critical Pair Lemma 25

The following expressions are equivalent:

`(x1||x2||x3) == (x1||x2||x3||x1)`

Proof

Note that the input for the rule:

`x1_||x2_||(x3_&&x1_)→x1||x2`

contains a subpattern of the form:

`x3_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 23 and Substitution Lemma 15 respectively.

Critical Pair Lemma 26

The following expressions are equivalent:

`(x1||x2||x3)==(x1||x2||x2||x3)`

Proof

Note that the input for the rule:

`x1_||(x2_&&x1_)||x3_→x1||x3`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x2_||x1_)→x1`

where these rules follow from Critical Pair Lemma 24 and Substitution Lemma 14 respectively.

Critical Pair Lemma 27

The following expressions are equivalent:

`(x1||x2||x3)==(x3||x1||x1||x2)`

Proof

Note that the input for the rule:

`x1_||x2_||x3_||x1_→x1||x2||x3`

contains a subpattern of the form:

`x1_||x2_||x3_||x1_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 25 and Equationalized Axiom 3 respectively.

Substitution Lemma 23

It can be shown that:

`(x1||x2||x3)==(x3||x1||x2)`

Proof

We start by taking Critical Pair Lemma 27, and apply the substitution:

`x1_||x2_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 26.

Substitution Lemma 24

It can be shown that:

$(!(x_2||y_1||z_0)||y_1||z_0||x_2) == (\underset{\cdot}{a}_0||!\underset{\cdot}{a}_0)$

Proof

PROOF

We start by taking Equationalized Hypothesis 1, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 25

It can be shown that:

$(!(x_2||y_1||z_0)||z_0||x_2||y_1) == (a_0||!a_0)$

PROOF

We start by taking Substitution Lemma 24, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 26

It can be shown that:

$(!(x_2||z_0||y_1)||z_0||x_2||y_1) == (a_0||!a_0)$

PROOF

We start by taking Substitution Lemma 25, and apply the substitution:

x1_||x2_→x2||x1

which follows from Equationalized Axiom 3.

Substitution Lemma 27

It can be shown that:

$(!(x_2||z_0||y_1)||z_0||x_2||y_1) == (y_1||!y_1)$

PROOF

We start by taking Substitution Lemma 26, and apply the substitution:

x1_||!x1_→y_1||!y_1

which follows from Substitution Lemma 21.

Substitution Lemma 28

It can be shown that:

$(!(x_2||z_0||y_1)||z_0||x_2||y_1) == (y_1||!y_1)$

PROOF

We start by taking Substitution Lemma 27, and apply the substitution:

x1_→x1

which follows from Substitution Lemma 13.

Substitution Lemma 29

It can be shown that:

$(!(x_2||z_0||y_1)||z_0||x_2||y_1) == (y_1||!y_1)$

PROOF

We start by taking Substitution Lemma 28, and apply the substitution:

`x1_→x1`

which follows from Substitution Lemma 13.

Substitution Lemma 30

It can be shown that:

$(!(x_2||z_0||y_1)||y_1||z_0||x_2)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 29, and apply the substitution:

`x1_||x2_||x3_→x3||x1||x2`

which follows from Substitution Lemma 23.

Substitution Lemma 31

It can be shown that:

$(!(x_2||z_0||y_1)||z_0||y_1||x_2)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 30, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 32

It can be shown that:

$(!(x_2||z_0||y_1)||x_2||z_0||y_1)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 31, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 33

It can be shown that:

$(!(x_2||y_1||z_0)||x_2||z_0||y_1)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 32, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 34

It can be shown that:

$(!(x_2||y_1||z_0)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 33, and apply the substitution:

We start by taking Substitution Lemma 33, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 35

It can be shown that:

$(!(y_1||z_0||x_2)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 34, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 36

It can be shown that:

$(!(z_0||y_1||x_2)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 35, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 37

It can be shown that:

$(!(x_2||z_0||y_1)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 36, and apply the substitution:

`x1_||x2_||x3_→x3||x1||x2`

which follows from Substitution Lemma 23.

Substitution Lemma 38

It can be shown that:

$(!(y_1||x_2||z_0)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 37, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 39

It can be shown that:

$(!(y_1||z_0||x_2)||x_2||y_1||z_0)==(y_1||!y_1)$

Proof

We start by taking Substitution Lemma 38, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Substitution Lemma 40

It can be shown that:

```
(x2||y1||z0||!(y1||z0||x2))==(y1||!y1)
```

PROOF

We start by taking Substitution Lemma 39, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Substitution Lemma 41

It can be shown that:

```
(x2||y1||z0||!(y1||z0||x2))==(y1||!y1)
```

PROOF

We start by taking Substitution Lemma 40, and apply the substitution:

```
x1_→x1
```

which follows from Substitution Lemma 13.

Substitution Lemma 42

It can be shown that:

```
(x2||y1||z0||!(z0||x2||y1))==(y1||!y1)
```

PROOF

We start by taking Substitution Lemma 41, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Substitution Lemma 43

It can be shown that:

```
(x2||y1||z0||!(z0||x2||y1))==(y1||!y1)
```

PROOF

We start by taking Substitution Lemma 42, and apply the substitution:

```
x1_→x1
```

which follows from Substitution Lemma 13.

Substitution Lemma 44

It can be shown that:

```
(x2||y1||z0||!(y1||z0||x2))==(y1||!y1)
```

PROOF

We start by taking Substitution Lemma 43, and apply the substitution:

```
x1_||x2_||x3_→x3||x1||x2
```

which follows from Substitution Lemma 23.

Substitution Lemma 45

It can be shown that:

$(x_2 || y_1 || z_0 || ! (x_2 || y_1 || z_0)) == (y_1 || ! y_1)$

Proof

We start by taking Substitution Lemma 44, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Conclusion 1

We obtain the conclusion:

`True`

Proof

Take Substitution Lemma 45, and apply the substitution:

$x1\_ || ! x1\_ \to y_1 || ! y_1$

which follows from Critical Pair Lemma 8.

APPENDIX 6. Proof of PM implies CN1.

In[24]:=
```
proofPMimpCN1["ProofNotebook"]
```

Axiom 1

We are given that:

`PMAxioms`

Hypothesis 1

We would like to show that:

$\forall_{\{x,y,z\}}((x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)))$

Equationalized Axiom 1

We generate the "equationalized" axiom:

`x1==(x1||(x2&&!x2))`

Equationalized Axiom 2

We generate the "equationalized" axiom:

`x1==(x1&&(x2||!x2))`

Equationalized Axiom 3

We generate the "equationalized" axiom:

`(x1||x2)==(x2||x1)`

Equationalized Axiom 4

We generate the "equationalized" axiom:

`(x1||(x2&&x3))==((x1||x2)&&(x1||x3))`

Equationalized Axiom 5

We generate the "equationalized" axiom:

`((x1&&x2)||(x1&&x3))==(x1&&(x2||x3))`

Equationalized Axiom 6

We generate the "equationalized" axiom:

$(a_0 || !a_0) ==$ `PMAxioms`

Equationalized Axiom 7

We generate the "equationalized" axiom:

`(x1&&x2)==(x2&&x1)`

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!(!x_2||y_1)||!(!y_1||z_0)||!x_2||z_0) == (a_0||!a_0)$

Critical Pair Lemma 1

The following expressions are equivalent:

`(x1||(x2&&x3))==((x2||x1)&&(x1||x3))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Equationalized Axiom 4 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 2

The following expressions are equivalent:

`x1==(x1&&PMAxioms)`

Proof

Note that the input for the rule:

`x1_&&(x2_||!x2_)→x1`

contains a subpattern of the form:

`x2_||!x2_`

which can be unified with the input for the rule:

a_0`||!`a_0`→PMAxioms`

where these rules follow from Equationalized Axiom 2 and Equationalized Axiom 6 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`x1==(PMAxioms&&x1)`

Proof

Note that the input for the rule:

`x1_&&PMAxioms→x1`

contains a subpattern of the form:

`x1_&&PMAxioms`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 2 and Equationalized Axiom 7 respectively.

Critical Pair Lemma 4

The following expressions are equivalent:

`(x1&&(PMAxioms||x2))==(x1||(x1&&x2))`

Proof

Note that the input for the rule:

```
(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)
```

contains a subpattern of the form:

```
x1_&&x2_
```

which can be unified with the input for the rule:

```
x1_&&PMAxioms→x1
```

where these rules follow from Equationalized Axiom 5 and Critical Pair Lemma 2 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

```
(x1||!x1)==PMAxioms
```

Proof

Note that the input for the rule:

```
PMAxioms&&x1_→x1
```

contains a subpattern of the form:

```
PMAxioms&&x1_
```

which can be unified with the input for the rule:

```
x1_&&(x2_||!x2_)→x1
```

where these rules follow from Critical Pair Lemma 3 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

```
x1==(x1||!PMAxioms)
```

Proof

Note that the input for the rule:

```
x1_||(x2_&&!x2_)→x1
```

contains a subpattern of the form:

```
x2_&&!x2_
```

which can be unified with the input for the rule:

```
PMAxioms&&x1_→x1
```

where these rules follow from Equationalized Axiom 1 and Critical Pair Lemma 3 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

```
(x1||(!x1&&x2))==(PMAxioms&&(x1||x2))
```

Proof

Note that the input for the rule:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

contains a subpattern of the form:

```
x1_||x2_
```

which can be unified with the input for the rule:

`x1_||!x1_→PMAxioms`

where these rules follow from Equationalized Axiom 4 and Critical Pair Lemma 5 respectively.

Substitution Lemma 1

It can be shown that:

`(x1||(!x1&&x2))==(x1||x2)`

Proof

We start by taking Critical Pair Lemma 7, and apply the substitution:

`PMAxioms&&x1_→x1`

which follows from Critical Pair Lemma 3.

Critical Pair Lemma 8

The following expressions are equivalent:

`x1==(!PMAxioms||x1)`

Proof

Note that the input for the rule:

`x1_||!PMAxioms→x1`

contains a subpattern of the form:

`x1_||!PMAxioms`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`(x1&&!x1)==!PMAxioms`

Proof

Note that the input for the rule:

`!PMAxioms||x1_→x1`

contains a subpattern of the form:

`!PMAxioms||x1_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 8 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 10

The following expressions are equivalent:

`(x1&&(!x1||x2))==(!PMAxioms||(x1&&x2))`

Proof

Note that the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&!x1_→!PMAxioms`

where these rules follow from Equationalized Axiom 5 and Critical Pair Lemma 9 respectively.

Substitution Lemma 2

It can be shown that:

`(x1&&(!x1||x2))==(x1&&x2)`

Proof

We start by taking Critical Pair Lemma 10, and apply the substitution:

`!PMAxioms||x1_→x1`

which follows from Critical Pair Lemma 8.

Critical Pair Lemma 11

The following expressions are equivalent:

`(x1&&(PMAxioms||!x1))==(x1||!PMAxioms)`

Proof

Note that the input for the rule:

`x1_||(x1_&&x2_)→x1&&(PMAxioms||x2)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&!x1_→!PMAxioms`

where these rules follow from Critical Pair Lemma 4 and Critical Pair Lemma 9 respectively.

Substitution Lemma 3

It can be shown that:

`(x1&&(PMAxioms||!x1))==x1`

Proof

We start by taking Critical Pair Lemma 11, and apply the substitution:

`x1_||!PMAxioms→x1`

which follows from Critical Pair Lemma 6.

Critical Pair Lemma 12

The following expressions are equivalent:

`(x1&&(PMAxioms||x2))==(x1||(x2&&x1))`

Proof

Note that the input for the rule:

`x1_||(x1_&&x2_)→x1&&(PMAxioms||x2)`

`x1_||(x1_&&x2_)→x1&&(PMAxioms||x2)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 7 respectively.

Critical Pair Lemma 13

The following expressions are equivalent:

`(x1||x1)==(x1&&(PMAxioms||!x1))`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&x1_)→x1&&(PMAxioms||x2)`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 12 respectively.

Substitution Lemma 4

It can be shown that:

`(x1||x1)==x1`

Proof

We start by taking Critical Pair Lemma 13, and apply the substitution:

`x1_&&(PMAxioms||!x1_)→x1`

which follows from Substitution Lemma 3.

Critical Pair Lemma 14

The following expressions are equivalent:

`(x1||PMAxioms)==(x1||!x1)`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&PMAxioms→x1`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 2 respectively.

Substitution Lemma 5

It can be shown that:

`(x1||PMAxioms)==PMAxioms`

```
(x1||PMAxioms)==PMAxioms
```

Proof

We start by taking Critical Pair Lemma 14, and apply the substitution:

```
x1_||!x1_→PMAxioms
```

which follows from Critical Pair Lemma 5.

Critical Pair Lemma 15

The following expressions are equivalent:

```
(x1||x1)==(x1||!PMAxioms)
```

Proof

Note that the input for the rule:

```
x1_||(!x1_&&x2_)→x1||x2
```

contains a subpattern of the form:

```
!x1_&&x2_
```

which can be unified with the input for the rule:

```
x1_&&!x1_→!PMAxioms
```

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 9 respectively.

Substitution Lemma 6

It can be shown that:

```
(x1||x1)==x1
```

Proof

We start by taking Critical Pair Lemma 15, and apply the substitution:

```
x1_||!PMAxioms→x1
```

which follows from Critical Pair Lemma 6.

Critical Pair Lemma 16

The following expressions are equivalent:

```
(x1||(x2&&x1))==((x1||x2)&&x1)
```

Proof

Note that the input for the rule:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

contains a subpattern of the form:

```
x1_||x3_
```

which can be unified with the input for the rule:

```
x1_||x1_→x1
```

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 4 respectively.

Substitution Lemma 7

It can be shown that:

```
(x1&&(PMAxioms||x2))==((x1||x2)&&x1)
```

Proof

We start by taking Critical Pair Lemma 16, and apply the substitution:

`x1_||(x2_&&x1_)→x1&&(PMAxioms||x2)`

which follows from Critical Pair Lemma 12.

Critical Pair Lemma 17

The following expressions are equivalent:

`PMAxioms==(PMAxioms||x1)`

Proof

Note that the input for the rule:

`x1_||PMAxioms→PMAxioms`

contains a subpattern of the form:

`x1_||PMAxioms`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 5 and Equationalized Axiom 3 respectively.

Substitution Lemma 8

It can be shown that:

`x1_||(x2_&&x1_)→x1&&PMAxioms`

Proof

We start by taking Critical Pair Lemma 12, and apply the substitution:

`PMAxioms||x1_→PMAxioms`

which follows from Critical Pair Lemma 17.

Substitution Lemma 9

It can be shown that:

`x1_||(x2_&&x1_)→x1`

Proof

We start by taking Substitution Lemma 8, and apply the substitution:

`x1_&&PMAxioms→x1`

which follows from Critical Pair Lemma 2.

Substitution Lemma 10

It can be shown that:

`(x1&&PMAxioms)==((x1||x2)&&x1)`

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

`PMAxioms||x1_→PMAxioms`

which follows from Critical Pair Lemma 17.

Substitution Lemma 11

It can be shown that:

`x1 == ((x1||x2)&&x1)`

Proof

We start by taking Substitution Lemma 10, and apply the substitution:

`x1_&&PMAxioms→x1`

which follows from Critical Pair Lemma 2.

Substitution Lemma 12

It can be shown that:

`x1 == (x1&&(x1||x2))`

Proof

We start by taking Substitution Lemma 11, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 7.

Critical Pair Lemma 18

The following expressions are equivalent:

`x1 == (x1&&(x2||x1))`

Proof

Note that the input for the rule:

`x1_&&(x1_||x2_)→x1`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 12 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 19

The following expressions are equivalent:

`(x1||x2) == (x1||x2||x1)`

Proof

Note that the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[168,x1_&&PMAxiom`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Substitution Lemma 9 and Substitution Lemma 12 respectively.

Critical Pair Lemma 20

Critical Pair Lemma 20

The following expressions are equivalent:

`x1⩵(x1&&x1)`

PROOF

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x2_||x1_`

which can be unified with the input for the rule:

`x1_||x1_→x1`

where these rules follow from Critical Pair Lemma 18 and Substitution Lemma 6 respectively.

Critical Pair Lemma 21

The following expressions are equivalent:

`(x1&&x2)⩵(x1&&x2&&x2)`

PROOF

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x2_||x1_`

which can be unified with the input for the rule:

`Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[168,x1_&&PMAxion`

where these rules follow from Critical Pair Lemma 18 and Substitution Lemma 9 respectively.

Critical Pair Lemma 22

The following expressions are equivalent:

`(x1||x2||!x1)⩵(x1||!x1)`

PROOF

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x2_||x1_)→x1`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 18 respectively.

Substitution Lemma 13

It can be shown that:

`(x1||x2||!x1)⩵PMAxioms`

PROOF

We start by taking Critical Pair Lemma 22, and apply the substitution:

```
x1_||!x1_→PMAxioms
```

which follows from Critical Pair Lemma 5.

Substitution Lemma 14

It can be shown that:

```
(x1||x2)==(x1||x1||x2)
```

Proof

We start by taking Critical Pair Lemma 19, and apply the substitution:

```
x1_||x2_→x2||x1
```

which follows from Equationalized Axiom 3.

Critical Pair Lemma 23

The following expressions are equivalent:

```
(x1||((x1||x2)&&x3))==((x1||x2)&&(x1||x3))
```

Proof

Note that the input for the rule:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

contains a subpattern of the form:

```
x1_||x2_
```

which can be unified with the input for the rule:

```
x1_||x1_||x2_→x1||x2
```

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 14 respectively.

Substitution Lemma 15

It can be shown that:

```
(x1||((x1||x2)&&x3))==(x1||(x2&&x3))
```

Proof

We start by taking Critical Pair Lemma 23, and apply the substitution:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

which follows from Equationalized Axiom 4.

Critical Pair Lemma 24

The following expressions are equivalent:

```
(x1||(x2&&(x1||x3)))==((x1||x2)&&(x1||x3))
```

Proof

Note that the input for the rule:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

contains a subpattern of the form:

```
x1_||x3_
```

which can be unified with the input for the rule:

Out[24]=

```
x1_||x1_||x2_→x1||x2
```

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 14 respectively.

Substitution Lemma 16

It can be shown that:

```
(x1||(x2&&(x1||x3)))==(x1||(x2&&x3))
```

Proof

We start by taking Critical Pair Lemma 24, and apply the substitution:

```
(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)
```

which follows from Equationalized Axiom 4.

Substitution Lemma 17

It can be shown that:

```
x1==(x1&&x1)
```

Proof

We start by taking Critical Pair Lemma 20, and apply the substitution:

```
x1_&&x2_→x2&&x1
```

which follows from Equationalized Axiom 7.

Substitution Lemma 18

It can be shown that:

```
(x1&&x2)==(x2&&x1&&x2)
```

Proof

We start by taking Critical Pair Lemma 21, and apply the substitution:

```
x1_&&x2_→x2&&x1
```

which follows from Equationalized Axiom 7.

Critical Pair Lemma 25

The following expressions are equivalent:

```
(x1&&x2&&!x1)==(x1&&!x1)
```

Proof

Note that the input for the rule:

```
x1_&&(!x1_||x2_)→x1&&x2
```

contains a subpattern of the form:

```
!x1_||x2_
```

which can be unified with the input for the rule:

```
Language`EquationalProofDump`getConstructRule[EquationalProof`ApplyLemma[168,x1_&&PMAxion
```

where these rules follow from Substitution Lemma 2 and Substitution Lemma 9 respectively.

Substitution Lemma 19

It can be shown that:

`(x1&&x2&&!x1)==!PMAxioms`

Proof

We start by taking Critical Pair Lemma 25, and apply the substitution:

`x1_&&!x1_→!PMAxioms`

which follows from Critical Pair Lemma 9.

Critical Pair Lemma 26

The following expressions are equivalent:

`(x1&&x1)==(x1&&PMAxioms)`

Proof

Note that the input for the rule:

`x1_&&(!x1_||x2_)→x1&&x2`

contains a subpattern of the form:

`!x1_||x2_`

which can be unified with the input for the rule:

`x1_||!x1_→PMAxioms`

where these rules follow from Substitution Lemma 2 and Critical Pair Lemma 5 respectively.

Substitution Lemma 20

It can be shown that:

`x1==(x1&&PMAxioms)`

Proof

We start by taking Critical Pair Lemma 26, and apply the substitution:

`x1_&&x1_→x1`

which follows from Substitution Lemma 17.

Substitution Lemma 21

It can be shown that:

`True`

Proof

We start by taking Substitution Lemma 20, and apply the substitution:

`x1_&&PMAxioms→x1`

which follows from Critical Pair Lemma 2.

Critical Pair Lemma 27

The following expressions are equivalent:

`PMAxioms==(!x1||x2||x1)`

Proof

Note that the input for the rule:

`x1_||x2_||!x1_→PMAxioms`

contains a subpattern of the form:

`!x1_`

which can be unified with the input for the rule:

`x1_→x1`

where these rules follow from Substitution Lemma 13 and Substitution Lemma 21 respectively.

Critical Pair Lemma 28

The following expressions are equivalent:

`(!PMAxioms) == (!x1&&x2&&x1)`

Proof

Note that the input for the rule:

`x1_&&x2_&&!x1_→!PMAxioms`

contains a subpattern of the form:

`!x1_`

which can be unified with the input for the rule:

`x1_→x1`

where these rules follow from Substitution Lemma 19 and Substitution Lemma 21 respectively.

Critical Pair Lemma 29

The following expressions are equivalent:

`(!PMAxioms) == (!(x1||x2)&&x1)`

Proof

Note that the input for the rule:

`!x1_&&x2_&&x1_→!PMAxioms`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 28 and Substitution Lemma 12 respectively.

Substitution Lemma 22

It can be shown that:

`(!PMAxioms) == (x1&&!(x1||x2))`

Proof

We start by taking Critical Pair Lemma 29, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 7.

Critical Pair Lemma 30

The following expressions are equivalent:

`(x1||!(!x1||x2)) == (x1||!PMAxioms)`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&!(x1_||x2_)→!PMAxioms`

where these rules follow from Substitution Lemma 1 and Substitution Lemma 22 respectively.

Substitution Lemma 23

It can be shown that:

`(x1||!(!x1||x2))==x1`

Proof

We start by taking Critical Pair Lemma 30, and apply the substitution:

`x1_||!PMAxioms→x1`

which follows from Critical Pair Lemma 6.

Critical Pair Lemma 31

The following expressions are equivalent:

`(!(!x1||x2))==(!(!x1||x2)&&x1)`

Proof

Note that the input for the rule:

`x1_&&(x2_||x1_)→x1`

contains a subpattern of the form:

`x2_||x1_`

which can be unified with the input for the rule:

`x1_||!(!x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 18 and Substitution Lemma 23 respectively.

Critical Pair Lemma 32

The following expressions are equivalent:

`(x1||(x2&&(x2||x3)))==(x1||x2||(x1&&x3))`

Proof

Note that the input for the rule:

`x1_||((x1_||x2_)&&x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`(x1_||x2_)&&x3_`

which can be unified with the input for the rule:

`(x1_||x2_)&&(x2_||x3_)→x2||(x1&&x3)`

where these rules follow from Substitution Lemma 15 and Critical Pair Lemma 1 respectively.

Substitution Lemma 24

It can be shown that:

`(x1||x2) == (x1||x2||(x1&&x3))`

Proof

We start by taking Critical Pair Lemma 32, and apply the substitution:

`x1_&&(x1_||x2_)→x1`

which follows from Substitution Lemma 12.

Critical Pair Lemma 33

The following expressions are equivalent:

`(x1||x2) == (x1||(x1&&x3)||x2)`

Proof

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x2_||(x1_&&x3_)`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 24 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 34

The following expressions are equivalent:

`(x1||x2) == (x1||x2||(x3&&x1))`

Proof

Note that the input for the rule:

`x1_||x2_||(x1_&&x3_)→x1||x2`

contains a subpattern of the form:

`x1_&&x3_`

which can be unified with the input for the rule:

`x1_&&x2_&&x1_→x2&&x1`

where these rules follow from Substitution Lemma 24 and Substitution Lemma 18 respectively.

Critical Pair Lemma 35

The following expressions are equivalent:

`(x1||x2) == (x1||(x3&&x1)||x2)`

Proof

Note that the input for the rule:

`x1_||(x1_&&x2_)||x3_→x1||x3`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x2_&&x1_→x2&&x1`

where these rules follow from Critical Pair Lemma 33 and Substitution Lemma 18 respectively.

Critical Pair Lemma 36

The following expressions are equivalent:

`(x1||x2||x3) == (x1||x2||x3||x1)`

Proof

Note that the input for the rule:

`x1_||x2_||(x3_&&x1_)→x1||x2`

contains a subpattern of the form:

`x3_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Critical Pair Lemma 34 and Substitution Lemma 12 respectively.

Critical Pair Lemma 37

The following expressions are equivalent:

`(x1||x2||x3) == (x1||x2||x2||x3)`

Proof

Note that the input for the rule:

`x1_||(x2_&&x1_)||x3_→x1||x3`

contains a subpattern of the form:

`x2_&&x1_`

which can be unified with the input for the rule:

`x1_&&(x2_||x1_)→x1`

where these rules follow from Critical Pair Lemma 35 and Critical Pair Lemma 18 respectively.

Substitution Lemma 25

It can be shown that:

`(!(!x1||x2)) == (x1&&!(!x1||x2))`

Proof

We start by taking Critical Pair Lemma 31, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 7.

Critical Pair Lemma 38

The following expressions are equivalent:

`(x1||x2||x3) == (x3||x1||x1||x2)`

Proof

Note that the input for the rule:

`x1_||x2_||x3_||x1_→x1||x2||x3`

contains a subpattern of the form:

`x1_||x2_||x3_||x1_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 36 and Equationalized Axiom 3 respectively.

Substitution Lemma 26

It can be shown that:

`(x1||x2||x3) == (x3||x1||x2)`

Proof

We start by taking Critical Pair Lemma 38, and apply the substitution:

`x1_||x2_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 37.

Critical Pair Lemma 39

The following expressions are equivalent:

`True`

Proof

Note that the input for the rule:

`x1_||x2_||x3_↔x3_||x1_||x2_`

contains a subpattern of the form:

`x1_||x2_||x3_`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 26 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 40

The following expressions are equivalent:

`(x1||x2||!(x1||x2)) == PMAxioms`

Proof

Note that the input for the rule:

`x1_||x2_||x3_→x1||x2||x3`

contains a subpattern of the form:

`x1_||x2_||x3_`

which can be unified with the input for the rule:

`x1_||!x1_→PMAxioms`

where these rules follow from Critical Pair Lemma 39 and Critical Pair Lemma 5 respectively.

Critical Pair Lemma 41

The following expressions are equivalent:

```
(x1&&(x2||!(!x1||x2)))==(x1&&PMAxioms)
```

PROOF

Note that the input for the rule:

```
x1_&&(!x1_||x2_)→x1&&x2
```

contains a subpattern of the form:

```
!x1_||x2_
```

which can be unified with the input for the rule:

```
x1_||x2_||!(x1_||x2_)→PMAxioms
```

where these rules follow from Substitution Lemma 2 and Critical Pair Lemma 40 respectively.

Substitution Lemma 27

It can be shown that:

```
(x1&&(x2||!(!x1||x2)))==x1
```

PROOF

We start by taking Critical Pair Lemma 41, and apply the substitution:

```
x1_&&PMAxioms→x1
```

which follows from Critical Pair Lemma 2.

Critical Pair Lemma 42

The following expressions are equivalent:

```
(x1||(x2&&!(!x2||x1)))==(x1||x2)
```

PROOF

Note that the input for the rule:

```
x1_||(x2_&&(x1_||x3_))→x1||(x2&&x3)
```

contains a subpattern of the form:

```
x2_&&(x1_||x3_)
```

which can be unified with the input for the rule:

```
x1_&&(x2_||!(!x1_||x2_))→x1
```

where these rules follow from Substitution Lemma 16 and Substitution Lemma 27 respectively.

Substitution Lemma 28

It can be shown that:

```
(x1||!(!x2||x1))==(x1||x2)
```

PROOF

We start by taking Critical Pair Lemma 42, and apply the substitution:

```
x1_&&!(!x1_||x2_)→!(!x1||x2)
```

which follows from Substitution Lemma 25.

Substitution Lemma 29

It can be shown that:

$(!(!x_2||y_1)||!x_2||z_0||!(!y_1||z_0)) == (a_0||!a_0)$

PROOF

We start by taking Equationalized Hypothesis 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 30

It can be shown that:

$(!x_2||z_0||!(!y_1||z_0)||!(!x_2||y_1)) == (a_0||!a_0)$

PROOF

We start by taking Substitution Lemma 29, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 31

It can be shown that:

$(!x_2||z_0||!(!y_1||z_0)||!(!x_2||y_1)) == PMAxioms$

PROOF

We start by taking Substitution Lemma 30, and apply the substitution:

$a_0||!a_0 \to PMAxioms$

which follows from Equationalized Axiom 6.

Substitution Lemma 32

It can be shown that:

$(!x_2||z_0||!(!y_1||z_0)||!(!x_2||y_1)) == PMAxioms$

PROOF

We start by taking Substitution Lemma 31, and apply the substitution:

`x1_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 39.

Substitution Lemma 33

It can be shown that:

$(!x_2||z_0||!(!y_1||z_0)||!(!x_2||y_1)) == PMAxioms$

PROOF

We start by taking Substitution Lemma 32, and apply the substitution:

`x1_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 39.

Substitution Lemma 34

It can be shown that:

$(!x_2||z_0||y_1||!(!x_2||y_1))$ ==PMAxioms

Proof

We start by taking Substitution Lemma 33, and apply the substitution:

`x1_||!(!x2_||x1_)→x1||x2`

which follows from Substitution Lemma 28.

Substitution Lemma 35

It can be shown that:

$(!x_2||y_1||z_0||!(!x_2||y_1))$ ==PMAxioms

Proof

We start by taking Substitution Lemma 34, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 36

It can be shown that:

$(!x_2||y_1||z_0||!(y_1||!x_2))$ ==PMAxioms

Proof

We start by taking Substitution Lemma 35, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 37

It can be shown that:

$(!x_2||!(y_1||!x_2)||y_1||z_0)$ ==PMAxioms

Proof

We start by taking Substitution Lemma 36, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 38

It can be shown that:

$(!(y_1||!x_2)||y_1||z_0||!x_2)$ ==PMAxioms

Proof

We start by taking Substitution Lemma 37, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 39

It can be shown that:

$(!(!x_2||y_1)||y_1||z_0||!x_2)$ ==PMAxioms

PROOF

We start by taking Substitution Lemma 38, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 40

It can be shown that:

$(!(!x_2||y_1)||y_1||z_0||!x_2)==\text{PMAxioms}$

PROOF

We start by taking Substitution Lemma 39, and apply the substitution:

`x1_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 39.

Substitution Lemma 41

It can be shown that:

$(!(!x_2||y_1)||y_1||z_0||!x_2)==\text{PMAxioms}$

PROOF

We start by taking Substitution Lemma 40, and apply the substitution:

`x1_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 39.

Substitution Lemma 42

It can be shown that:

$(!(!x_2||y_1)||z_0||!x_2||y_1)==\text{PMAxioms}$

PROOF

We start by taking Substitution Lemma 41, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 43

It can be shown that:

$(!(y_1||!x_2)||z_0||!x_2||y_1)==\text{PMAxioms}$

PROOF

We start by taking Substitution Lemma 42, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 44

It can be shown that:

$(!(y_1||!x_2)||z_0||!x_2||y_1)==\text{PMAxioms}$

PROOF

We start by taking Substitution Lemma 43, and apply the substitution:

`x1_||x2_||x3_→x1||x2||x3`

which follows from Critical Pair Lemma 39.

Substitution Lemma 45

It can be shown that:

$(!(y_1||!x_2)||z_0||y_1||!x_2)==PMAxioms$

PROOF

We start by taking Substitution Lemma 44, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Conclusion 1

We obtain the conclusion:

`True`

PROOF

Take Substitution Lemma 45, and apply the substitution:

`!x1_||x2_||x1_→PMAxioms`

which follows from Critical Pair Lemma 27.

APPENDIX 7. Proof of PM implies CN2.

In[25]:= `proofPMimpCN2["ProofNotebook"]`

Axiom 1

We are given that:

`PMAxioms`

Hypothesis 1

We would like to show that:

$\forall_{\{x,y\}} (x \Rightarrow (!x \Rightarrow y))$

Equationalized Axiom 1

We generate the "equationalized" axiom:

`x1 == (x1 || (x2 && !x2))`

Equationalized Axiom 2

We generate the "equationalized" axiom:

`x1 == (x1 && (x2 || !x2))`

Equationalized Axiom 3

We generate the "equationalized" axiom:

`(x1 || x2) == (x2 || x1)`

Equationalized Axiom 4

We generate the "equationalized" axiom:

`(x1 || (x2 && x3)) == ((x1 || x2) && (x1 || x3))`

Equationalized Axiom 5

We generate the "equationalized" axiom:

`((x1 && x2) || (x1 && x3)) == (x1 && (x2 || x3))`

Equationalized Axiom 6

We generate the "equationalized" axiom:

$(a_0 \,||\, !a_0) == \text{PMAxioms}$

Equationalized Axiom 7

We generate the "equationalized" axiom:

`(x1 && x2) == (x2 && x1)`

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!x_1 \,||\, x_1 \,||\, y_0) == (a_0 \,||\, !a_0)$

Critical Pair Lemma 1

Critical Pair Lemma 1

The following expressions are equivalent:

`x1==(x1&&PMAxioms)`

PROOF

Note that the input for the rule:

`x1_&&(x2_||!x2_)→x1`

contains a subpattern of the form:

`x2_||!x2_`

which can be unified with the input for the rule:

`a0||!a0→PMAxioms`

where these rules follow from Equationalized Axiom 2 and Equationalized Axiom 6 respectively.

Critical Pair Lemma 2

The following expressions are equivalent:

`x1==(PMAxioms&&x1)`

PROOF

Note that the input for the rule:

`x1_&&PMAxioms→x1`

contains a subpattern of the form:

`x1_&&PMAxioms`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 7 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`(x1&&(PMAxioms||x2))==(x1||(x1&&x2))`

PROOF

Note that the input for the rule:

`(x1_&&x2_)||(x1_&&x3_)→x1&&(x2||x3)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&PMAxioms→x1`

where these rules follow from Equationalized Axiom 5 and Critical Pair Lemma 1 respectively.

Critical Pair Lemma 4

The following expressions are equivalent:

`(x1||!x1)==PMAxioms`

PROOF

Note that the input for the rule:

`PMAxioms&&x1_→x1`

contains a subpattern of the form:

`PMAxioms&&x1_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 2 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

`x1==(x1||!PMAxioms)`

Proof

Note that the input for the rule:

`x1_||(x2_&&!x2_)→x1`

contains a subpattern of the form:

`x2_&&!x2_`

which can be unified with the input for the rule:

`PMAxioms&&x1_→x1`

where these rules follow from Equationalized Axiom 1 and Critical Pair Lemma 2 respectively.

Critical Pair Lemma 6

The following expressions are equivalent:

`(x1||(!x1&&x2))==(PMAxioms&&(x1||x2))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||!x1_→PMAxioms`

where these rules follow from Equationalized Axiom 4 and Critical Pair Lemma 4 respectively.

Substitution Lemma 1

It can be shown that:

`(x1||(!x1&&x2))==(x1||x2)`

Proof

We start by taking Critical Pair Lemma 6, and apply the substitution:

`PMAxioms&&x1_→x1`

which follows from Critical Pair Lemma 2.

Critical Pair Lemma 7

The following expressions are equivalent:

`x1==(!PMAxioms||x1)`

PROOF

Note that the input for the rule:

`x1_||!PMAxioms→x1`

contains a subpattern of the form:

`x1_||!PMAxioms`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 5 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1&&!x1)==!PMAxioms`

PROOF

Note that the input for the rule:

`!PMAxioms||x1_→x1`

contains a subpattern of the form:

`!PMAxioms||x1_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 7 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 9

The following expressions are equivalent:

`(x1&&(PMAxioms||!x1))==(x1||!PMAxioms)`

PROOF

Note that the input for the rule:

`x1_||(x1_&&x2_)→x1&&(PMAxioms||x2)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&!x1_→!PMAxioms`

where these rules follow from Critical Pair Lemma 3 and Critical Pair Lemma 8 respectively.

Out[25]=

Substitution Lemma 2

It can be shown that:

`(x1&&(PMAxioms||!x1))==x1`

PROOF

We start by taking Critical Pair Lemma 9, and apply the substitution:

`x1_||!PMAxioms→x1`

which follows from Critical Pair Lemma 5.

Critical Pair Lemma 10

The following expressions are equivalent:

`(x1&&(PMAxioms||x2))==(x1||(x2&&x1))`

Proof

Note that the input for the rule:

`x1_||(x1_&&x2_)→x1&&(PMAxioms||x2)`

contains a subpattern of the form:

`x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 3 and Equationalized Axiom 7 respectively.

Critical Pair Lemma 11

The following expressions are equivalent:

`(x1||x1)==(x1&&(PMAxioms||!x1))`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`x1_||(!x1_&&x2_)`

which can be unified with the input for the rule:

`x1_||(x2_&&x1_)→x1&&(PMAxioms||x2)`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 10 respectively.

Substitution Lemma 3

It can be shown that:

`(x1||x1)==x1`

Proof

We start by taking Critical Pair Lemma 11, and apply the substitution:

`x1_&&(PMAxioms||!x1_)→x1`

which follows from Substitution Lemma 2.

Critical Pair Lemma 12

The following expressions are equivalent:

`(x1||PMAxioms)==(x1||!x1)`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&PMAxioms→x1`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 1 respectively.

Substitution Lemma 4

It can be shown that:

`(x1||PMAxioms)==PMAxioms`

Proof

We start by taking Critical Pair Lemma 12, and apply the substitution:

`x1_||!x1_→PMAxioms`

which follows from Critical Pair Lemma 4.

Critical Pair Lemma 13

The following expressions are equivalent:

`(x1||(x2&&x1))==((x1||x2)&&x1)`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x3_`

which can be unified with the input for the rule:

`x1_||x1_→x1`

where these rules follow from Equationalized Axiom 4 and Substitution Lemma 3 respectively.

Substitution Lemma 5

It can be shown that:

`(x1&&(PMAxioms||x2))==((x1||x2)&&x1)`

Proof

We start by taking Critical Pair Lemma 13, and apply the substitution:

`x1_||(x2_&&x1_)→x1&&(PMAxioms||x2)`

which follows from Critical Pair Lemma 10.

Critical Pair Lemma 14

The following expressions are equivalent:

`PMAxioms==(PMAxioms||x1)`

Proof

Note that the input for the rule:

`x1_||PMAxioms→PMAxioms`

contains a subpattern of the form:

`x1_||PMAxioms`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Substitution Lemma 4 and Equationalized Axiom 3 respectively.

Substitution Lemma 6

It can be shown that:

`(x1&&PMAxioms)==((x1||x2)&&x1)`

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

`PMAxioms||x1_→PMAxioms`

which follows from Critical Pair Lemma 14.

Substitution Lemma 7

It can be shown that:

`x1==((x1||x2)&&x1)`

Proof

We start by taking Substitution Lemma 6, and apply the substitution:

`x1_&&PMAxioms→x1`

which follows from Critical Pair Lemma 1.

Substitution Lemma 8

It can be shown that:

`x1==(x1&&(x1||x2))`

Proof

We start by taking Substitution Lemma 7, and apply the substitution:

`x1_&&x2_→x2&&x1`

which follows from Equationalized Axiom 7.

Critical Pair Lemma 15

The following expressions are equivalent:

`(x1||!x1||x2)==(x1||!x1)`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&(x1_||x2_)→x1`

where these rules follow from Substitution Lemma 1 and Substitution Lemma 8 respectively.

where these rules follow from Substitution Lemma [illegible] and Substitution Lemma [illegible] respectively.

Substitution Lemma 9

It can be shown that:

```
(x1||!x1||x2)==PMAxioms
```

Proof

We start by taking Critical Pair Lemma 15, and apply the substitution:

```
x1_||!x1_→PMAxioms
```

which follows from Critical Pair Lemma 4.

Substitution Lemma 10

It can be shown that:

$(!x_1||x_1||y_0)==PMAxioms$

Proof

We start by taking Equationalized Hypothesis 1, and apply the substitution:

$\underset{\bullet}{a}_0||!\underset{\bullet}{a}_0→PMAxioms$

which follows from Equationalized Axiom 6.

Conclusion 1

We obtain the conclusion:

```
True
```

Proof

Take Substitution Lemma 10, and apply the substitution:

```
x1_||!x1_||x2_→PMAxioms
```

which follows from Substitution Lemma 9.

APPENDIX 8. Proof of PM implies CN3.

In[26]:= proofPMimpCN3["ProofNotebook"]

Axiom 1

We are given that:

PMAxioms

Hypothesis 1

We would like to show that:

$\forall_x((!x \Rightarrow x) \Rightarrow x)$

Equationalized Axiom 1

We generate the "equationalized" axiom:

x1 == (x1 || (x2 && !x2))

Equationalized Axiom 2

We generate the "equationalized" axiom:

x1 == (x1 && (x2 || !x2))

Equationalized Axiom 3

We generate the "equationalized" axiom:

(x1 || x2) == (x2 || x1)

Equationalized Axiom 4

We generate the "equationalized" axiom:

(x1 || (x2 && x3)) == ((x1 || x2) && (x1 || x3))

Equationalized Axiom 5

We generate the "equationalized" axiom:

$(a_0 || !a_0)$ == PMAxioms

Equationalized Axiom 6

We generate the "equationalized" axiom:

(x1 && x2) == (x2 && x1)

Equationalized Hypothesis 1

We generate the "equationalized" hypothesis:

$(!(x_0 || x_0) || x_0) == (a_0 || !a_0)$

Critical Pair Lemma 1

The following expressions are equivalent:

x1 == (x1 && PMAxioms)

PROOF

PROOF

Note that the input for the rule:

`x1_&&(x2_||!x2_)→x1`

contains a subpattern of the form:

`x2_||!x2_`

which can be unified with the input for the rule:

`a0||!a0→PMAxioms`

where these rules follow from Equationalized Axiom 2 and Equationalized Axiom 5 respectively.

Critical Pair Lemma 2

The following expressions are equivalent:

`x1==(PMAxioms&&x1)`

PROOF

Note that the input for the rule:

`x1_&&PMAxioms→x1`

contains a subpattern of the form:

`x1_&&PMAxioms`

which can be unified with the input for the rule:

`x1_&&x2_↔x2_&&x1_`

where these rules follow from Critical Pair Lemma 1 and Equationalized Axiom 6 respectively.

Critical Pair Lemma 3

The following expressions are equivalent:

`(x1||!x1)==PMAxioms`

PROOF

Note that the input for the rule:

`PMAxioms&&x1_→x1`

contains a subpattern of the form:

`PMAxioms&&x1_`

which can be unified with the input for the rule:

`x1_&&(x2_||!x2_)→x1`

where these rules follow from Critical Pair Lemma 2 and Equationalized Axiom 2 respectively.

Critical Pair Lemma 4

The following expressions are equivalent:

`x1==(x1||!PMAxioms)`

PROOF

Note that the input for the rule:

`x1_||(x2_&&!x2_)→x1`

contains a subpattern of the form:

`x2_&&!x2_`

`x2_&&!x2_`

which can be unified with the input for the rule:

`PMAxioms&&x1_→x1`

where these rules follow from Equationalized Axiom 1 and Critical Pair Lemma 2 respectively.

Critical Pair Lemma 5

The following expressions are equivalent:

`(x1||(!x1&&x2))==(PMAxioms&&(x1||x2))`

Proof

Note that the input for the rule:

`(x1_||x2_)&&(x1_||x3_)→x1||(x2&&x3)`

contains a subpattern of the form:

`x1_||x2_`

which can be unified with the input for the rule:

`x1_||!x1_→PMAxioms`

where these rules follow from Equationalized Axiom 4 and Critical Pair Lemma 3 respectively.

Out[26]=

Substitution Lemma 1

It can be shown that:

`(x1||(!x1&&x2))==(x1||x2)`

Proof

We start by taking Critical Pair Lemma 5, and apply the substitution:

`PMAxioms&&x1_→x1`

which follows from Critical Pair Lemma 2.

Critical Pair Lemma 6

The following expressions are equivalent:

`x1==(!PMAxioms||x1)`

Proof

Note that the input for the rule:

`x1_||!PMAxioms→x1`

contains a subpattern of the form:

`x1_||!PMAxioms`

which can be unified with the input for the rule:

`x1_||x2_↔x2_||x1_`

where these rules follow from Critical Pair Lemma 4 and Equationalized Axiom 3 respectively.

Critical Pair Lemma 7

The following expressions are equivalent:

`(x1&&!x1)==!PMAxioms`

Proof

Note that the input for the rule:

`!PMAxioms||x1_→x1`

contains a subpattern of the form:

`!PMAxioms||x1_`

which can be unified with the input for the rule:

`x1_||(x2_&&!x2_)→x1`

where these rules follow from Critical Pair Lemma 6 and Equationalized Axiom 1 respectively.

Critical Pair Lemma 8

The following expressions are equivalent:

`(x1||x1)==(x1||!PMAxioms)`

Proof

Note that the input for the rule:

`x1_||(!x1_&&x2_)→x1||x2`

contains a subpattern of the form:

`!x1_&&x2_`

which can be unified with the input for the rule:

`x1_&&!x1_→!PMAxioms`

where these rules follow from Substitution Lemma 1 and Critical Pair Lemma 7 respectively.

Substitution Lemma 2

It can be shown that:

`(x1||x1)==x1`

Proof

We start by taking Critical Pair Lemma 8, and apply the substitution:

`x1_||!PMAxioms→x1`

which follows from Critical Pair Lemma 4.

Substitution Lemma 3

It can be shown that:

$(!(x_0||x_0)||x_0)==(a_0||!a_0)$

Proof

We start by taking Equationalized Hypothesis 1, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 4

It can be shown that:

$(x_0||!(x_0||x_0))==(a_0||!a_0)$

Proof

We start by taking Substitution Lemma 3, and apply the substitution:

We start by taking Substitution Lemma 3, and apply the substitution:

`x1_||x2_→x2||x1`

which follows from Equationalized Axiom 3.

Substitution Lemma 5

It can be shown that:

`(x0||!(x0||x0))==PMAxioms`

Proof

We start by taking Substitution Lemma 4, and apply the substitution:

`a0||!a0→PMAxioms`

which follows from Equationalized Axiom 5.

Substitution Lemma 6

It can be shown that:

`(x0||!x0)==PMAxioms`

Proof

We start by taking Substitution Lemma 5, and apply the substitution:

`x1_||x1_→x1`

which follows from Substitution Lemma 2.

Conclusion 1

We obtain the conclusion:

`True`

Proof

Take Substitution Lemma 6, and apply the substitution:

`x1_||!x1_→PMAxioms`

which follows from Critical Pair Lemma 3.

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